THE BÔCHER MEMORIAL PRIZE

THE FIRST AWARD. REGULATIONS GOVERNING THE SECOND AND THIRD AWARDS.

An interesting and significant event in the annals of the Society was the awarding at the Annual Meeting of 1923 of the first Bôcher Memorial Prize to George David Birkhoff, Professor of Mathematics at Harvard University. The memoir so crowned with the first prize ever given under the auspices of the Society is entitled *Dynamical systems with two degrees of freedom* and appeared in the TRANSACTIONS OF THIS SOCIETY, volume 18, pages 199–300.

The presentation was made at the morning session on December 28, by Professor H. S. White, a member of the committee appointed by the Council to make recommendations concerning the first award. After summing up in happy fashion the history of the foundation of the prize, Professor White gave a brief analysis of the memoir. Professor Birkhoff, in acknowledging the honor done him, spoke of Bôcher's contribution to the mathematical ideals and traditions of the Society.

The Committee on Award, consisting of Professors C. N. Haskins (chairman), T. S. Fiske, and H. S. White, were unanimous in their recommendation of this memoir. They pointed out, however, in presenting their report to the Council, that owing to the delay in printing the TRANSACTIONS, the instructions given them (see this BULLETIN, vol. 28, p. 42) were not self-consistent; the Council then voted to revise and supplement the regulations and to make a second award at the Annual Meeting in 1924.

The revised regulations are as follows:

(1) The Bôcher Memorial Prize of $100 shall from time to time be awarded for a notable research memoir, published

in the TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY by a resident of the United States or Canada.

(2) The age of the recipient of the prize shall not be over forty years, and the prize shall not be awarded twice to the same person.

(3) The age and place of residence of a writer shall be reckoned as of the first day of the month of date of issue of the number of the TRANSACTIONS containing the article in question, or the first part of it.

(4) The second award, to be announced at the Annual Meeting of 1924, shall be made with reference to volumes 19 to 24 inclusive.

(5) The third award, to be announced at the Annual Meeting of 1928, shall be made with reference to the volumes beginning with that issued for the year 1923 and ending with that issued for the year 1927.

R. G. D. RICHARDSON,
Secretary.

SPECIAL MEETING OF THE SAN FRANCISCO SECTION

A special meeting of the San Francisco Section of the American Mathematical Society was held at the University of Washington, Seattle, on Saturday, December 22, 1923. The following seventeen members of the Society were in attendance:

Bell, Daniel Buchanan, A. F. Carpenter, DeCou, Feinler, Gavett, Griffin, Griffiths, Kent, McAlister, W. E. Milne, Moritz, Mullemeister, Neikirk, Smail, Stager, Winger.

Professor A. F. Carpenter, chairman of the Section, presided.

Titles and abstracts of the papers presented are listed below. Professor Cajori's paper was read by title, while

the paper of Mr. Kelly and Miss Small was read by Professor Griffin, who introduced these authors to the Society.

1. Professor E. T. Bell: *On certain quinary quadratic forms.*

This paper has appeared in the March-April number of this BULLETIN.

2. Professor Daniel Buchanan: *Asymptotic satellites near the straight line equilibrium points (elliptic case).*

If two finite masses move in ellipses about their common center of gravity, there are five points, called *equilibrium points* or *points of libration,* at which an infinitesimal body will remain fixed, relatively to the moving bodies, if given proper initial projections. Three of these points are collinear and lie on the line joining the centers of gravity of the finite bodies. The other two lie in the plane of motion of the finite bodies at the vertices of the equilateral triangles described on the join of their centers of gravity.

The author determines orbits which approach asymptotically (1) the straight line equilibrium points, and (2) the two- and three-dimensional periodic orbits near these straight line equilibrium points which were obtained by Moulton in his *Periodic Orbits,* Chapter VII. The asymptotic orbits are somewhat similar to those obtained by the author in the corresponding circular case (AMERICAN JOURNAL, vol. 41, pp. 79–110).

3. Professor Daniel Buchanan: *Periodic and asymptotic satellites near the equilateral triangle equilibrium points (elliptic case).*

This paper deals with the periodic and asymptotic orbits near the equilateral triangle points of libration as the finite masses move in ellipses. Two- and three-dimensional periodic orbits are first obtained and later the orbits which are asymptotic to the three-dimensional orbits and to the points of libration themselves. As in the circular case determined by Buck (Moulton, *Periodic Orbits,* Chapter IX) the two-dimensional periodic orbits exist only when the ratio of the finite masses is small. The orbits, however, which are asymptotic to the equilibrium points exist, as in the corresponding circular case (Buchanan, TRANSACTIONS

OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY, vol. 22, No. 15, pp. 309–340) only when the ratio of the finite masses is larger than in the case of the periodic orbits. Owing to the variation in this ratio, it has not been possible according to the methods employed in this paper to obtain orbits asymptotic to the two-dimensional periodic orbits.

4. Professor A. F. Carpenter: *Complex cones and cone-cubics of a ruled surface.*

In this paper four quadric cones are associated with each line of a ruled surface by means of the osculating linear complex. Their intersections, pair at a time, determine four space cubics, called primary cone-cubics, two of which lie upon the osculating quadric. Each of these cubics is shown to be projectively equivalent to the primary flecnode cubic. By an analogous process four secondary cone-cubics are defined. These bear to the primary cone-cubics the relation borne by the secondary flecnode cubic to the primary flecnode cubic. A number of theorems involving these curves are proved.

5. Mr. F. J. Feinler: *On the Bernoulli numbers.*

Adams calculated the Bernoulli numbers up to B_{124} by the help of the von Staudt and Clausen Theorem, requiring the tedious handling of fractions. There is no practical necessity to recalculate this series of valuable constants, except perhaps for checking Adams' results. Yet there remains always the theoretical interest in the genesis and the relations of these numbers. The denominators are found to present a very simple law for the reappearance of the prime number series. But if another slightly modified formula is used for the denominators, the tedious handling of fractions is altogether avoided. The values thus obtained are then reduced by the quotient of the two denominator formulas. The resulting series is found identical with that of Adams.

6. Miss Lois Griffiths: *Contact curves of the rational plane cubic.*

This account of contact curves of the rational plane cubic developed from consideration of the subject of involution as applied to rational curves. The determinate case is that in which the complete intersection of the contact curve of order n and the cubic is two distinct points, one a d-point contact p^d and the other a complementary contact p^{3n-d}.

Representative theorems for curves of this type are (1) All points of the cubic are related in sets of $3n$ each; within a single set d points are of one type, contacts p^d, and the remaining $(3n-d)$ are contacts p^{3n-d}, such that each of the $(3n-d)$ points is a complementary contact for each of the d points, and for no other point of the cubic, and vice versa. Thus all contact curves are related in sets of $d(3n-d)$ each. (2) Superimposed on each set of $d(3n-d)$ contact curves, when and only when n is even, is a non-contact curve of order n, cutting the cubic singly in each of the $3n$ contacts for that set of contact curves.

7. Professor F. C. Kent: *Note on the derivation and reduction of annuity formulas.*

The annuity formulas in which the nominal rate of j is convertible m times a year and the annuity is paid p times a year are reduced to simple standard forms which are readily adapted to computation by means of ordinary annuity tables. Also a simplified method is presented for deriving the standard formulas for the amount and present value of annuities certain.

8. Professor L. L. Smail: *A theorem on convergent factors in summable series.*

By making use of certain results of James (ANNALS OF MATHEMATICS, vol. 21, p. 123) and Schur (JOURNAL FÜR MATHEMATIK, vol. 151, pp. 79–111) on the general theory of summability, a theorem is obtained on convergence factors in series summable (C, 1).

9. Professor Florian Cajori: *The growth of legend about Sir Isaac Newton.*

The author considers the growth of legend in accounts of the discovery of the law of gravitation, of the invention of the reflecting telescope, of the delay in the publication of the *Principia,* of action at a distance, and of the wave theory of light.

10. Mr. C. T. Kelly and Miss Dana Small: *Curves conjugate to conics with respect to lines and circles* (*Type* I).

The authors discuss in some detail the conjugate of given conics with respect to a straight line or circle as the basic

curve, in the case where the relation between radii vectores is of the form $\varrho_c = \varrho_g = \varrho_b$ or $2\varrho_r$.

13. Professor R. M. Winger: *Note on the invariants of the ternary icosahedral group.*

The fundamental system of invariants of this group, given by Klein, *Ikosaeder*, includes a conic A and a set of six lines, say F. There is thus a pencil of invariant sextics $A^3 + \lambda F = 0$ which contains Klein's curve B ($\lambda = -1$), the author's rational sextic R ($\lambda = 27/5$), AMERICAN JOURNAL, 1916, and the sextic invariant of the Valentiner group (Wiman). The salient properties of the pencil are discussed. In particular it is shown that Klein's C which is a rational curve of order ten and class six is the projective dual of R.

14. Professor L. I. Neikirk: *Some non-associative algebras.*

Several examples of these finite linear algebras are given. Left handed division is always possible (except by zero) and is unique. Right-handed division is not always possible and when possible is not always unique. There may be one, two or four quotients. Some of these algebras have nilpotent elements, also elements with equations of the type $S^3 = S^2$. Here S^3 is a left-handed power and $S^3 = [S\cdot(S\cdot S)]$. Multiplication is distributive on the right and non-distributive on the left in some cases. Multiplication tables are given.

R. M. WINGER,
Secretary for the special meeting.

THE THIRTIETH ANNUAL MEETING OF THE SOCIETY

The thirtieth annual meeting of the American Mathematical Society was held at Columbia University on Thursday and Friday, December 27–28, 1923. Dormitory accommodations were furnished by Columbia University and Barnard College, and attending members were introduced at the Faculty Club. About seventy-five members attended the dinner on Thursday evening, at which Professor Coolidge spoke on the progress of the Endowment Fund. At the close of the sessions it was voted to express the thanks of the Society to Columbia University for its courtesy, continued through more than thirty years. It was voted to thank also the Governors of the Faculty Club, and the Committee on Arrangements for the meeting, which consisted of Professors T. S. Fiske (chairman), G. W. Mullins, and L. P. Siceloff, Dr. G. A. Pfeiffer, and Mr. A. E. Meder.

The attendance included the following ninety-five members of the Society:

C. R. Adams, Alexander, C. R. Ballantine, J. P. Ballantine, Barnum, Bernstein, Birkhoff, Borden, R. W. Burgess, Burwell, Clements, Abraham Cohen, Coolidge, C. F. Craig, Dadourian, Douglas, Edmondson, Eisenhart, Fields, Fine, Fite, Forsyth, Philip Franklin, Fry, Gafafer, Gilman, Glenn, P. H. Graham, Grant, M. C. Graustein, W. C. Graustein, Hammond, W. L. Hart, Harter, Hawkes, Hazlett, Robert Henderson, Hille, Himwich, Hotelling, Huntington, Joffe, Kasner, Kazarinoff, Lamson, Langman, Leib, Linehan, Lynch, McGiffert, MacInness, Mathews, Meder, Mirick, H. H. Mitchell, Frank Morley, Marston Morse, Mullins, Oglesby, Olds, F. W. Owens, Pell, Post, Rainich, Ranum, Raynor, Reddick, R. G. D. Richardson, Ritt, Seely, Seidlin, Shook, Siceloff, Simmons, Slotnick, Simons, C. E. Smith, D. E. Smith, W. M. Smith, Virgil Snyder, M. H. Stone, Elijah Swift, J. S. Taylor, Tracey, Tyler, Veblen, Evelyn Walker, J. L. Walsh, Wedderburn, Weisner, Weiss, H. S. White, E. E. Whitford, Whittemore, R. G. Wood.

The Council elected the following sixty-five persons to membership in the Society:

Professor Olafur T. Anderson, Wesley College;
Professor William Elijah Anderson, Miami University;
Dr. Bernardo Ig. Baidaff, Buenos Aires;
Dr. John Perry Ballantine, Columbia University;
Professor Maurice Baudin, Miami University;

Professor James Marcus Bledsoe, East Texas State Teachers College;
Mr. William Miller Bond, College of the City of New York;
Mr. Orley Edward Brown, Northwestern University;
Dr. Keivin Burns, Allegheny Observatory;
Mr. William Elmer Byrne, Rensselaer Polytechnic Institute;
Professor Theodosia Tucker Calloway, Stephens Junior College;
Professor George B. Clinkscales, Converse College;
Dean James Frost Cross, St. John's College;
Professor Samuel Ridgely Cruse, University of Arizona;
Dr. Heber Doust Curtis, Allegheny Observatory;
Mr. Alfred Davis, Soldan High School, St. Louis;
Mr. Hannibal A. Davis, West Virginia University;
Professor Samuel Dickstein, University of Warsaw;
Mr. Walter Anthony Dynan, Amityville, N. Y.;
Mr. George Gilbert Entz, Los Angeles;
Mr. Elgin George Fassel, Cleveland Life Insurance Company;
Mr. James Hampton Fithian, Brown University;
Mr. Anselm J. Fleisig, St. Procopius College;
Professor Robert Dale Ford, St. Lawrence University;
Miss Alice Ann Grant, Brown University;
Professor Ruby Mabel Grimes, Northern Normal and Industrial School, Aberdeen, S. D.;
Mr. Duncan Claire Harkins, West Virginia University;
Professor Eli Stuart Haynes, University of Missouri;
Professor Chester Rodney Hillard, College of the Ozarks;
Professor Frederick Chapman Jonah, Acadia University;
Mr. Arthur Daniel Kinsman, Ipswich, Mass.;
Professor Carl Dewey Laws, Hampden-Sidney College;
Dean Thomas Jefferson Leslie, Austin College;
Mr. Locke Ellsworth Lunn, State High School, Heron Lake, Minn.;
Mr. Arthur Louis McCarty, Lowell High School, San Francisco;
Dean Lee Marcus McCoy, Morgan College;
Professor Raymond Earl Manchester, Kent State College;
Mr. Alfred Joseph Maria, Rice Institute;
Professor Gertrude Whittier Mendenhall, North Carolina College for Women;
Mr. Alfred William Muldrew, St. John's Technical High School, Winnipeg;
Mr. James A. Nyswander, University of Chicago;
Dr. Robert Bowie Owens, Franklin Institute;
Mr. George Ambrose Parkinson, University of Wisconsin;
Professor Frederick Malling Pedersen, College of the City of New York;
Miss Sallie E. Pero, American Telephone and Telegraph Company;
Professor Otto J. Ramler, Catholic University;
Professor James Nelson Rice, Catholic University;
Mr. Harold E. Riter, Daniel McIntyre Collegiate Institute, Winnipeg;
Mr. William Haldeman Rittenhouse, Philadelphia;
Miss Georgia Edna Robinson, Stephens College;
Mr. Harry Mance Roeser, Bureau of Standards;
Mr. James Shohat, University of Chicago;
Dr. Charles Albert Shook, Yale University;
Professor Robert Frederic Smith, College of the City of New York;
Dean Julia Spalding, Christian College;
Mr. Arthur A. Sperling, University of Wisconsin, Milwaukee Branch;
Mr. Robert Meldrum Stewart, Dominon Observatory, Ottawa;
Professor Robert Ambrose Thornton, Shaw University;
Professor Ernest R. Tucker, Texas Christian University;

Professor Charles Ambrose Van Velzer, Carthage College;
Mr. Albert Huntoon Wait, University of Wisconsin;
Miss Fannie Watkins, Winthrop College;
Mr. Lawrence E. Widmark, Star Electric Motor Company;
Mr. Edgar William Woolard, United States Weather Bureau;
Mr. Floyd Eugene Young, Oregon Agricultural College.

The total membership of the Society is now 1250, including 83 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 541; the number of papers read was 266. The number of members attending at least one meeting was 331. At the annual election 172 votes were cast.

The report of the Treasurer was received, showing a balance of $9,638.78, exclusive of special funds; of this, $5,726.91 is reserved to secure the life memberships. Sales of the Society's publications during the year amounted to $3,570.03. The report of the auditors (Mr. S. A. Joffe and Professor W. J. Berry) was deferred until the February meeting.

The Board of Trustees adopted a budget for 1924.

The librarian reported that the Library of the Society now contains 6,546 volumes.

At the annual election, which closed on Thursday afternoon, the following trustees and officers and other members of the Council were chosen:

Trustees, Professor G. A. BLISS, Professor W. B. FITE, Mr. ROBERT HENDERSON, Professor R. G. D. RICHARDSON, Professor OSWALD VEBLEN.

Vice-Presidents, Professor E. V. HUNTINGTON (one year); Professors T. H. HILDEBRANDT and J. H. M. WEDDERBURN (two years).

Secretary, Professor R. G. D. RICHARDSON.

Assistant Secretary, Professor ARNOLD DRESDEN.

Treasurer, Professor W. B. FITE.

Librarian, Professor R. C. ARCHIBALD.

Member of the Editorial Committee of the Bulletin, Professor ARNOLD DRESDEN.

Member of the Editorial Committee of the Transactions, Professor EDWARD KASNER.

Members of the Council, Professor H. W. TYLER (one year); Professor J. L. COOLIDGE (two years); Professors HARRY BATEMAN, G. A. BLISS, SOLOMON LEFSCHETZ, C. L. E. MOORE, ANNA J. PELL (three years).

The tellers appointed by President Veblen to count the ballots were Professor R. E. Gilman, Dr. Jesse Douglas, and Mr. A. E. Meder.

An account of the proceedings connected with the presentation of the first Bôcher Memorial Prize and of the regulations governing the second and third awards is contained elsewhere in this issue.

On account of proposed absence from the country, Professor E. B. Van Vleck presented his resignation as representative of the Society on the National Research Council. Professor H. F. Blichfeldt was selected to complete the term which expires on July 1, 1924, and also to represent the Society for the succeeding period of three years. Professors A. B. Coble, Arnold Dresden, L. E. Dickson, and Virgil Snyder were elected as additional representatives of the Society on the American Section of the International Mathematical Union.

A committee on the western meetings of the Society was appointed for a two-year period, consisting of Professors T. H. Hildebrandt (chairman), Henry Blumberg, and Arnold Dresden. Dean H. L. Hodgkins and Professor R. G. D. Richardson were appointed to represent the Society on the Council of the American Association for the Advancement of Science for the year 1924.

The committee on membership was continued, to consist of Professors Abraham Cohen (chairman), E. B. Stouffer, and C. E. Smith (permanent secretary), and such others as President Veblen may appoint.

A letter was read from the secretary of the Physico-Mathematical Society of Japan, thanking the Society for the message of sympathy sent in September and recounting some of the effects of the disaster on mathematical research in that country.

The Council recommended to the Society that the By-Laws be amended so as to include ex-Presidents in the Council for a period of six years after expiration of their presidential terms.

It was announced that Professor H. H. Mitchell had been appointed chairman of the Committee on the next Colloquium, in place of Professor L. E. Dickson resigned; and that Dean H. L. Hodgkins had represented the Society at the inauguration of President Lewis of George Washington University.

Announcement was made by President Veblen, who is also Chairman of the Division of Physical Sciences of the National Research Council, concerning the extension of one group of the Research Fellowships to include mathematics. This group of fellowships which is adminstered by the Research Fellowship Board of the National Research Council and supported by the Rockefeller Foundation now covers mathematics, physics, and chemistry; and, beginning July 1, 1925, the annual income is increased to $125,000. In order that adequate use be made of this remarkable addition to the resources for mathematical research, members are urged to give information concerning talented and well-equipped young men and women.

President Veblen presided at the sessions on Thursday, relieved by Vice-President Tyler; during the sessions on Friday, Vice-Presidents Huntington and Wedderburn alternated in the chair.

At the session on Friday morning a paper was read, at the request of the program committee, by Professor Virgil Snyder, on *Problems connected with involutorial transformations in space.* This paper appeared in full in the March-April issue of this BULLETIN.

Titles and abstracts of the papers read at this meeting follow below. The papers of Professor Bell, Dr. Zeldin, Professor Lipka, Dr. Pfeiffer, Dr. Wiener, Professor Ford, Mr. Michal, and Professor Ritt, and the second papers of Professor Huntington, Dr. Walsh, Dr. Ballantine, and Dr. Taylor, were read by title.

1. Professor W. B. Fite: *The analytic functions defined by a certain functional differential equation.*

In this paper the author considers the solutions in the complex plane of the functional differential equation $y''(z) = A^2 y(\sin z) = 0$. He finds that they are entire functions of z. Some theorems relating to the existence and distribution of the real zeros of these functions are proved. Every such function has at least one real zero. Nothing is determined concerning the complex zeros.

2. Dr. K. W. Lamson: *The field equations of relativity as integrability conditions.*

The condition for the vanishing of the sum of the Riemann (two-dimensional) curvatures determined by μ mutually perpendicular directions in the general Riemann n-space is $K_\mu = 0$, where $K_\mu = \sum g_{ij}\, g_{ij}\, R_{ii,\,jj}$. Here $R_{ii,\,jj}$ is the Riemann-Christoffel 4-index symbol, and the i's are any μ numbers from the set $1, 2, \ldots, n$, and similarly for the j's. The summation is taken over all permutations of the i's among themselves and of the j's among themselves. The object of the paper is to prove that $K_\mu = 0$ is a necessary and sufficient condition for the existence of a contact transformation which transforms $\sum g^{ij}\, p_i\, p_j$ into the sum of squares. This contact transformation is determined by $n - \mu + 2$ relations between the old and new point coordinates. In particular, if $n = 4$, then $K_3 = 0$ is equivalent to the Einstein field equations. The general solution of the field equations is found by algebraic processes.

3. Dr. Jesse Douglas: *Characterization of spaces of constant curvature by the arrangement of their geodesics.*

Necessary conditions are derived that a family of curves in n-space defined by a system of differential equations of the form $d^2x_i/dx_1^2 = G_i\,(x_j,\, dx_j/dx_1)$ be such that through every three points a two-spread passes which contains ∞^2 curves of the family. As a special case, a proof is obtained of Schur's theorem (MATHEMATISCHE ANNALEN, 1886) that the existence of a geodesic two-spread through every three points of a Riemann space characterizes the space as having constant curvature. A generalization is developed to the effect that "two-spread" can be replaced by "k-spread".

4. Mr. G. Y. Rainich: *The non-linearity of Maxwell's equations in general relativity theory.*

The Maxwell equations consist of two sets of four equations each which have to be satisfied, respectively, by the covariant and mixed components of the electromagnetic tensor; for vacuum each set is linear. But these two sets are not sufficient to determine the electromagnetic field; we must consider together with them the equations which give the connection between the covariant and mixed components, and these equations involve the g's which are connected with the coefficients of the electromagnetic tensor by the energy relation which is not linear; therefore the whole system is non-linear. In order to be able to consider these things without introducing the restriction connected with the use of the g's, we introduce a new system of representation for curved space*. Each point is characterized by four coordinates as usual, but to represent vectors and tensors we introduce in every bundle cartesian coordinates. We have thus no distinction between co- and contravariant quantities, and the algebra of tensors is simplified. But additional complications arise in the tensor analysis, and they are responsible in this case for the non-linearity of Maxwell's equations.

5. Professor J. L. Coolidge: *A simple proof of Lüroth's theorem.*

Lüroth's theorem states that if x and y are rational functions of a parameter t, not both constants, the curve so defined has the deficiency zero. The author gives a new proof based on Zeuthen's correspondence formula, and communicates one by Professor Osgood based on abelian integrals.

6. Professor B. A. Bernstein: *Representation of three-element algebras.*

The author gives a method of obtaining a concrete representation of any binary operation or dyadic relation possible in an algebra consisting of three elements. The representations are all arithmetic and modular. The device used is an extension of that employed by him in the representation of two-element algebras.

* Cf. the author's paper in PROCEEDINGS OF THE NATIONAL ACADEMY, Dec., 1923.

7. Dr. J. L. Walsh: *On the expansion of analytic functions in series of polynomials.*

Let C be a simple closed finite analytic curve in the z-plane including in its interior the origin. Then there exist two sets of functions, $p_0(z), p_1(z), \ldots, p_n(z), \ldots$ and $q_1(z), q_2(z), \ldots, q_n(z), \ldots$, polynomials respectively in z and in $1/z$, such that if $f(z)$ be any function defined on C and satisfying on C a Lipschitz condition, then $f(z)$ can be expanded in the form $f(z) = a_0p_0(z) + a_1p_1(z) + \cdots + a_np_n(z) + \cdots + b_1q_1(z) + b_2q_2(z) + \cdots + b_nq_n(z) + \cdots$, where the former series converges uniformly in the closed region interior to C and the latter series converges uniformly in the closed region exterior to C and vanishes at infinity. The coefficients are given by formulas $a_k = \int_C f(z)\, s_k(z)\, dz$, $b_k = \int_C f(z)\, t_k(z) dz$, where the functions $s_k(z)$ and $t_k(z)$ depend on C but not on $f(z)$. The polynomial $p_k(z)$ has precisely k roots interior to C, and the polynomial $q_k(z)$ has precisely k roots exterior to C.

8. Professor E. V. Huntington: *A new set of postulates for betweenness, with proof of complete independence.*

A paper published in 1917 by E. V. Huntington and J. R. Kline (TRANSACTIONS OF THIS SOCIETY, vol. 18, pp. 301–325) gave eleven sets of independent postulates for betweenness, selected from a basic list of twelve postulates, A, B, C, D, 1–8. Of these eleven sets, three contained seven postulates each and eight contained six postulates each. In the present paper, a new postulate is added to the basic list, namely Postulate 9; it is then shown that the following set containing only five postulates, A, B, C, D, 9, is sufficient to define betweenness:

(A) If ABC, then CBA.

(B) If A, B, C are distinct, then at least one of the six permutations, $ABC, ACB, BAC, BCA, CAB, CBA$, is a true order.

(C) If ABC is true, then ACB is false.

(D) If ABC is true, then A, B, C are distinct.

(9) If A, B, C, X are distinct and ABC is true, then at least one of the orders ABX, XBC is true.

These five postulates are shown to be completely independent; and the long list of theorems in the earlier paper is extended so as to cover all the inter-relations among the thirteen postulates of the enlarged basic list. The paper will be offered to the TRANSACTIONS.

9. Professor E. V. Huntington: *A new set of completely independent postulates for cyclic order.*

If Postulate A for betweenness (see preceding abstract) is replaced by Postulate A′: If *ABC*, then *BCA*; then the five postulates A′, B, C, D, 9 will form a completely independent set of postulates for cyclic order. This paper is supplementary to two earlier papers on cyclic order, one presented on December 28, 1916 (see an abstract in the PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 2 (1916), pp. 630–631), and the other on October 27, 1923. It will be offered to the TRANSACTIONS.

10. Professor E. T. Bell: *Representations of integers in certain quadratic forms in* 2, 3, 4, 5 *indeterminates and allied class-number relations.*

When there is but one class in the principal genus of binary quadratic forms of a negative determinant, the number of representations of an integer in the form can be found at once in terms of the devisors of the integer. For certain ternary and quaternary forms there are similar simple theorems. For quinary forms the like exists in several cases if the integer represented is the square of an odd integer times a power of 2. All of the results give class number relations; those relating to quinary forms are of a wholly new type.

11. Dr. J. L. Walsh: *On Pellet's theorem concerning the roots of an algebraic equation.*

This paper furnishes a complete answer to the following question: If there are given the absolute values of the coefficients of an algebraic equation, what can be said of the regions of the complex plane in which the roots of the equation do or do not lie? The only statement which can be made regarding the roots is essentially a theorem proved by Pellet, DARBOUX'S BULLETIN, (2), vol. 5 (1881).

12. Dr. E. L. Post: *The mth derivative of a function of a function; calculus of mth derivatives.*

At the October, 1923, meeting of the Society the author presented a paper on a theory of generalized differentiation. The application of this to the Heaviside theory requires a usable knowledge of *m*th derivatives. The problem of

finding the mth derivative of a function of a function, $F(u)$, has always presented great difficulties. Thus, although the Faa di Bruno formula gives $(d^m/dx^m)\,F(u)$ as a finite series in $F^{(m)}(u)$ with coefficients depending wholly on u and its derivatives, the formula for these coefficients is quite unmanageable. The present paper concerns itself specifically with these coefficients, and sets up formulas for the coefficients of a sum, product, and quotient of two functions, and for a function of a function in terms of the coefficients of these functions. Thus in the last case if v is a function of u and u a function of x,

$$(m)v/(n)x = \sum_{l=m}^{n} ((m)v/(l)u)\,((l)u/(n)x),$$

where $(m)u/(n)x$ is the mth coefficient in the nth derivative of a function of u. These rules in connection with the coefficients for the simplest elementary functions may be said to constitute a calculus of the coefficients in question, and this in turn, added to the previously known formulas, gives a calculus of mth derivatives.

13. Professor W. C. Graustein: *Invariant directions in the theory of surfaces.*

Given a single surface or a map of one surface upon another, there exist directions, or vectors, which are relatively invariant under a change of parameters. It is the purpose of this paper to determine directions with this property and to demonstrate their usefulness in the theory of surfaces and the theory of maps.

14. Mr. D. K. Kazarinoff: *A property of the curvature of the line of intersection of two surfaces.*

Consider the two surfaces σ_1 and σ_2 and any point P of their line of intersection (σ_1, σ_2), where P is not a singular point for either of the surfaces. Consider at P the two planes τ_1 and τ_2 tangent to σ_1 and σ_2 respectively, and the plane curves of intersections (σ_1, τ_2) and (σ_2, τ_1). We regard the curvature of a curve as a vector of length equal to the absolute value of the curvature and directed as the principal normal from the curve to its center of curvature. We have, then, the following theorem, believed to be new: *The curvature of* (σ_1, σ_2) *at* P *is the geometric sum of the curvatures of* (σ_1, τ_2) *and* (σ_2, τ_1). A proof of remarkable simplicity, based on Meusnier's theorem, can be given.

15. Dr. Louis Weisner: *Simultaneous algebraic equations.*

Let two plane algebraic curves whose equations have their coefficients in a domain R intersect in a finite number of points all of which are in the finite part of the plane. It is shown that the group relative to R of the equation which the abscissas of the points of intersection of the two curves satisfy is the same as that which the ordinates satisfy. A similar theorem holds for any finite number of algebraic equations which are independent and consistent.

16. Professor Virgil Snyder: *On the types of monoidal involutions.*

The purpose of this paper is to unify the theory of monoidal involutions by proving all the known theorems by means of the same method. A number of new types are discussed, and it is shown that if irrational types exist they must be compounded from the perspective monoidal types, either by means of the plane Bertini type or the perspective Jonquières type. The involution which belongs to the cubic variety of four-dimensional space is included in the latter category.

17. Dr. Einar Hille: *On the zeros of the functions of the parabolic cylinder.*

The differential equation of the parabolic cylinder can be brought to one of the two forms $w'' \pm (z^2 - a^2)\, w = 0$, where a is real in general. The paper contains a study of the set of zeros and extrema of a given solution of one of these differential equations. The problem is considered from three different angles: (1) the asymptotic distribution of the set, (2) the geometric structure of the set, and (3) the variation of the structure with different initial conditions and with the parameter a.

18. Professor Edward Kasner: *Geodesic families and their generalizations.*

In this paper the author discusses various generalizations of the family of straight lines in ordinary n-space or geodesics in a Riemann space, in particular *cubic* families, *extremal* families, and *intersectional* families. Cubic families (which include the natural and velocity types previously studied by the author) are related to the geodesics of

Weyl's general affine connected space or the path geometry of Eisenhart and Veblen. Extremal families are related to the general integral of first order in the calculus of variations. Intersectional families are defined by the curves of intersection of families of surfaces or hypersurfaces (just as straight lines are obtained from planes). The overlapping of the various types is considered.

19. Dr. J. S. Taylor: *A statistical theory of depreciation.*

The principal results obtained are, first, a means of determining more accurately than heretofore the useful life of the average machine of a given type; second, a method of distributing the depreciation charges in such a way that the unit cost of output remains constant (or increases by given amounts); and third, a method of determining when an individual machine should be discarded when that time arrives in which the only conjecture is in the estimates of the costs for a new machine. Finally, the paper points out the interdependence of costs of output and the distribution of the depreciation charges, together with the fact that the useful life of a machine is affected by such distribution and by the current interest rate.

20. Dr. S. D. Zeldin: *A classification of integral invariants in the calculus of variations.*

The curve $y = f(x)$ which maximizes or minimizes the integral $I = \int F(x, y, y', y'', \ldots, y^{(n)})\, dx$ satisfies Lagrange's equation. The integral I is invariant under point and contact transformations. In this paper the author classifies the integrals I invariant under the contact transformations of the plane.

21. Professor Virgil Snyder: *Problems connected with involutorial transformations in space.*

This paper appeared in the March-April issue of this BULLETIN.

22. Professor Joseph Lipka: *Some theorems on trajectories in V_n.*

A study is made of a system of trajectories in a conservative field of force in space of n dimensions, taking, as a basis, n mutually orthogonal congruences and the Ricci coefficients of rotation $\gamma_{\alpha\beta h}$. Among other theorems,

the author proves that the sum of the squares of the geodesic curvatures of any n mutually orthogonal trajectories through a point is an invariant, *i. e.*, independent of the n directions chosen; this invariant is equal to $(n-1)$ times the square of the magnitude of the trajectory vector at the point.

23. Professor Joseph Lipka: *On irreversible dynamical systems under a zero force.*

These curves may be defined as the extremals of the variation problem $\int ds + \int d\Phi = \text{minimum}$, where ds^2 is the fundamental quadratic differential form in the space, and $d\Phi = \sum \Phi_i \, dx_i$, the Φ_i being functions of the coordinates. An interesting geometric property of these curves is the following: The locus of the centers of geodesic curvature of the ∞^1 curves of the system which pass through a given point is a hyperplane if the dimensionality of the space is *odd*; on the other hand, the above locus is a quartic hypersurface of a special variety if the dimensionality of the space is *even*. A curve of the system defined above may also be identified with Weyl's "Singularitäts-Kanal" in a field with a given structure.

24. Professor Marston Morse: *Relations between different types of critical points of a real analytic function of n real variables.*

If there be given a real analytic function of n variables, Birkhoff has shown that, under certain boundary hypotheses, the existence of m proper minima ($m > 1$) implies the existence of other critical points affording neither maxima nor minima. The present paper attacks the problem of determining the relations between general types of critical points each characterized by the signature of the quadratic form making up the terms of second order in the expansion of the given function in the neighborhood of the given critical point.

25. Dr. C. R. Adams: *The general theory of the linear q-difference equation.*

The author uses matrix methods in treating the system

$$g_i(qx, ry) = \sum_{j=1}^{n} p_{ij}(x, y)\, g_j(x, y), \quad (i = 1, 2, \ldots, n),$$

in which q and r are real or complex constants different from zero, the $p_{ij}(x, y)$ are polynomials in x and y, and the $g_i(x, y)$ are the functions to be determined; such a system is equivalent to a single homogeneous equation of the nth order. If $|q| \leqq 1, |r| \leqq 1$, the system possesses in general four matrices of formal series solutions (in powers of x and y, $1/x$ and $1/y$, x and $1/y$, $1/x$ and y). The case $|q|, |r| > 1$ is typical; then the elements of two matrices converge to analytic functions in the vicinities respectively of (0,0) and (∞, ∞). Two matrices of solutions are thus defined, one analytic whenever neither x nor y is infinite, and the other analytic save for poles whenever neither x nor y is zero. A matrix of periodic functions is defined by these two solutions, which, after transformation, appears as a matrix of triply periodic functions, a special case of a class of such functions discussed by Cousin. An inverse of the general existence theorem is proved and a method given for handling certain exceptional cases. It is further proved that if $|q| = |r| = 1$, there exists in general no analytic solution.

26. Professor O. F. Glenn: *The invariants of forms under the total group G_6, modulo* 2.

This paper appeared in full in the March-April number of this BULLETIN.

27. Professor Olive C. Hazlett: *On associated forms in the theory of formal modular covariants.*

By the aid of the symbolic theory of formal modular covariants, there is proved for such covariants a theorem which is closely analogous to Hermite's well known results for algebraic covariants. If h be any formal modular covariant of the binary form f with respect to the Galois field $GF[p^n]$ of order p^n, then (aside from a power of h) every formal modular covariant of f is expressible as a polynomial in the associated covariants with respect to h and a finite number of formal modular covariants obtained from h in a certain manner.

28. Professor Edward Kasner: *Intermediate curvatures in a Riemann space.*

A result analogous to Schur's theorem on Riemann curvature holds for Ricci curvature and for the intermediate

curvatures defined by the author in a previous paper: If any one of the curvatures is a point function (instead of an orientation function) it must be a constant. When this is proved for the Ricci curvature (by making use of Hilbert's identities) Schur's theorem is obtained as an immediate corollary, and by combining these results in a simple manner suggested by Mr. Rainich, the general theorem is obtained. When the curvatures are equated to zero, we obtain spaces which Dr. Lamson has integrated by means of contact transformations.

29. Dr. G. A. Pfeiffer: *A proof of Cauchy's integral theorem for any rectifiable boundary.*

Cauchy's integral theorem in the strong form, namely, if $f(z)$ is analytic in a domain bounded by any rectifiable curve and continuous on the set consisting of that domain and its boundary, then the integral of $f(z)$ over the boundary is zero, was established, so it seems, only recently. E. B. Van Vleck published in 1921 a proof of Green's lemma for any rectifiable boundary. The above theorem follows in the familiar way from that result. A direct proof of said theorem was published this year by S. Pollard, who in deriving certain preliminary results of a semi analysis situs character made use of some special results of de la Vallée Poussin. The present paper gives a proof of the same theorem, based only on well known mathematical facts. The lemma of the paper seems of interest on its own account. It states that the perimeters of all polygons which are within a domain S, bounded by a rectifiable curve, and which are derived from rectangular networks in a well known way, are less than $2\sqrt{2}$ times the length of the boundary of S.

30. Dr. J. P. Ballantine: *A generalization of the calculus of finite differences to include the differential calculus.*

In the calculus of finite differences, one speaks of a difference quotient taken at a set of distinct values. Differential calculus is often regarded as the limiting case when all the values approach equality. In this paper, it is shown that by properly enlarging the notion of difference quotient, differential calculus becomes a special, and not a limiting, case. Other special cases are found to be interesting, and have a practical application to the numerical solution of differential equations.

31. Dr. J. P. Ballantine: *A postulational introduction to the positive integers.*

Certain properties of the numbers 0 and 1, expressible without the use of any other numbers, are supposed. A class of undefined elements, called integers, is postulated, subject to a binary relation, called successor. The successor relation is in the form of a matrix, x'', whose values are 0 and 1 according as the second argument is not or is a successor of the first. It is subject to three axioms: Axiom 1, $S'(1' - S'x''1')1' = 0$; Axiom 2, $S'(1' - S'1'x'')1' = 1$; Axiom 3, $S'\xi'x'' = \xi' \cdot) \cdot \xi' = 0'$. It is proved that the successor of each integer is unique, that there exists (uniquely) a first integer, 1, that the set of successive successors of the integer 1 includes all integers except 1, and that the integers are linearly ordered.

32. Dr. J. S. Taylor: *A note on the theory of testimony.*

This paper deals with the question of determining the probability of the occurrence of a given event, whose a priori probability is known, when attested to by one or more observers. The *personal accuracy* of observation of an individual observer is defined as the limit of the accuracy of his observation as the a priori probability of the occurrence of an event approaches zero in a specified way. Given the personal accuracy of observation of an individual observer the probability of the occurrence of an event to which he attests can be determined for any case of a rather general type, and conversely. The results are then extended to the case of n observers. Finally, a method is obtained of determining the personal accuracy of observation of an individual observer to any degree of approximation, thus furnishing a means of obtaining the data upon which the foregoing results depend.

33. Professor C. H. Forsyth: *The determination of how much one would be justified in expending to extend the life and multiply the value of an article.*

Students of the mathematical theory of finance are familiar with the problem of determining how much one would be justified in expending to extend the life of an article a specified period, and the corresponding formula. In this paper the problem is extended and a formula derived for

determining how much one would be justified in expending to extend the life and at the same time multiply the value or productivity of the article. In fact, two formulas are derived so that a criterion of the extra expense can be determined whether both the life and the value of the article are increased or only one (either one) is increased and the other decreased.

34. Dr. Norbert Wiener: *A contribution to the theory of interpolation.*

Given a finite set of points S, surrounded by a curve C, the properties are developed of that function, harmonic on the interior of C, which assumes given values over S and has the least integral around C.

35. Professor L. R. Ford: *The numerical integration of differential equations.*

In solving a differential equation by Picard's method of successive approximations, the repeated integrations can be performed by the approximate methods of finite differences or by the use of a few terms of the Euler-Maclaurin expansion. Two questions arise: Does the numerical process converge? If so, what are the errors of the approximate solution? In this paper, it is proved that the process converges under quite general conditions, and a limit is found for the errors that result when the approximate method is repeated over successive intervals.

36. Mr. A. D. Michal: *Invariant functionals of closed plane curves under one-parameter groups of transformations of the plane.*

In this paper the author shows that there always exist functionals of closed plane curves invariant under a given arbitrary continuous one-parameter group of transformations of the plane. The functional of closed plane curves is assumed to be developable in a new type of expansion involving multiple double integrals, the region of integration being that of the area of the closed curves. The necessary and sufficient conditions for the invariance of the functionals in question reduce to the form of certain non-homogeneous linear partial differential equations of the first order. The symmetric solutions of these partial differential equations

yield the sought for integrands found in the assumed expansion of the invariant functionals in question. The actual calculation of the invariants is effected completely in the case of a number of well known continuous one-parameter groups of transformations of the plane.

37. Professor J. F. Ritt: *Note on Dirichlet series with complex exponents.*

This note gives a very simple proof of an extension made by E. Hille of a theorem of the present author's.

38. Professor E. L. Dodd: *Integration in certain n-dimensional fields with the aid of a general discontinuity factor.*

Use is made of $F(t)$, where $F(0) = 0$ and

$$\int_{-\infty}^{\infty} \frac{F(t)dt}{t} = K \neq 0, \neq \infty,$$

to effect integration of real functions in fields such as that determined by $-a < X < 0$, where X is a real function defined for all real values of its n variables. The general formula thus obtained, when applied to the case of $n = 1$, with $F(t) = \sin t$, and X linear, yields the Fourier integral equation. The discontinuity-factor method, used in the theory of probability to integrate $\varphi_1(x_1)\,\varphi_2(x) \ldots \varphi_n(x_n)$ between parallel planes, is shown to be valid in case each $\varphi_i(x_i)$ is bounded and integrable and absolutely integrable, and two of these functions are of limited variation. This device of integration, which makes integration proceed to ∞, while keeping the n variables separate, may then be used for almost any bounded function $\varphi_i(x_i)$ that has been suggested for frequency distributions, such as the Pearson types, finite Brun's series, Poisson exponential function, Charlier B-series, Jorgensen function, and Makeham life function.

R. G. D. RICHARDSON,
Secretary.

THE CINCINNATI MEETING OF THE SOCIETY

The twentieth Western meeting of the Society was held at the University of Cincinnati on Friday and Saturday, December 28 and 29, in conjunction with the meeting of the Mathematical Association of America and the seventy-fifth meeting of the American Association for the Advancement of Science.

The meeting was attended by about one hundred persons, among whom were the following seventy-nine members of the Society:

Alexander, R. B. Allen, Archibald, G. N. Armstrong, Atchison, Babb, Bareis, Barnett, E. M. Berry, Borger, Boyd Bradshaw, Brand, Cairns, Cajori, A. D. Campbell, Carmichael, Coble, H. H. Conwell, H. T. Davis, Davisson, Denton, Dickson, Dostal, Dresden, Edington, L. C. Emmons, Eversull, Focke, Fort, Gerst, Hancock, E. R. Hedrick, Holl, Dunham Jackson, Kindle, Kuhn, Lefschetz, Le Stourgeon, MacMillan, March, Marshall, T. E. Mason, G. A. Miller, C. N. Moore, R. L. Moore, C. C. Morris, Muehlman, Nassau, Olson, Overman, Palmié, Plant, Rasor, C. N. Reynolds, H. L. Rietz, E. D. Roe, Roever, Rothrock, Safford, Schottenfels, W. G. Simon, Sinclair, Slaught, Edwin R. Smith, I. W. Smith, Spenceley, R. B. Stone, Swartzel, B. M. Turner, Vandiver, Van Vleck, Veblen, J. H. Weaver, Norbert Wiener, Wilczewski, Yanney, J. W. Young, Zehring.

The session of Friday afternoon was held under the joint auspices of the Society, the Mathematical Association, and Section A of the American Association for the Advancement of Science, and was presided over by Professor Hancock, Chairman of Section A. The program consisted of the following three papers:

I. Professor G. A. Miller (as retiring Chairman of Section A): *American mathematics during three quarters of a century.*

II. Professor A. B. Coble (as retiring Chairman of the Chicago Section of the Society): *On the equation of the eighth degree.*

III. Professor L. E. Dickson (by invitation of the Program Committee): *Algebras and their arithmetics.*

At the close of this session Professor Jackson moved and those present unanimously adopted a resolution expressing to the University of Cincinnati and in particular to its

department of mathematics, their appreciation of the cordial reception and the hospitality accorded the visitors.

On Friday evening a dinner was held at the Hotel Gibson for mathematicians and astronomers, at which 110 persons were present. Mr. J. A. Shohat, who has recently come from Russia, expressed the appreciation of Russian scientists for the aid received from America. Professor Slaught spoke on the incorporation of the Society, and on the changes made in the By-Laws. Professor Dresden reported on the progress of the campaign for an endowment of $100,000. President Veblen, after supplementing the remarks of the previous speakers, announced the establishment by the National Research Council of Fellowships in Mathematics.

Professor Jackson reported for the committee appointed to make arrangements for a symposium at the meeting of April 1924, that the symposium lecture would be given by Professor H. L. Rietz on *The mathematical theory of statistics.*

At the session of Friday forenoon President Veblen took the chair. On Saturday Vice-President R. L. Moore presided, relieved by Professors A. B. Coble and R. C. Archibald.

The following papers were presented at the sessions of Friday morning, and Saturday morning and afternoon. Dr. Kraupner was introduced to the Society by Professor Hancock, Mr. Merriman by Professor C. N. Moore. Dr. Wilder's paper was presented by Professor R. L. Moore. The papers of Professor Reynolds, Dr. Kraupner, Professors Chittenden, MacDuffee, Wahlin, Cajori, Emch, and Sisam, Mr. Vandiver, Professor Smith, and Miss Schottenfels were read by title.

1. Professor W. E. Edington: *Theory of construction of group generators as substitutions.*

In general, to prove the existence of an abstract group generated by two or more operators obeying certain conditions requires the exposition of substitutions which fulfil all the required conditions on the generators. In this paper a theory for constructing such substitutions is discussed. Several theorems are proved giving general rules for finding substitutions fulfilling in certain cases the conditions

$s_1^m = s_2^n = (s_1 s_2)^k = 1$, and in some cases additional conditions. The theory of transpositions is made the basis of this theory.

2. Miss Bess M. Eversull: *On the summability of the triple Fourier series at points of discontinuity of the function involved.*

In a paper previously presented to the Society (this BULLETIN, vol. 28 (1922), p. 289) it was shown that if a function of three variables is finite and integrable (Lebesgue) its Fourier development is summable ($C1$) to the value of the function at all points within the region of continuity of the function. It is the purpose of the present paper to show by similar methods that the Fourier development of a function satisfying the above conditions is in certain cases summable ($C1$) at points of discontinuity of the function and to determine the values to which the series is summable. Results have been obtained where the point of discontinuity is such that all other points of discontinuity in the neighborhood lie on a plane or curved surface through that point, on two planes or curved surfaces having certain relative positions whose intersection passes through the point, and on three planes parallel to the coordinate planes whose common point of intersection is the point of discontinuity.

3. Mr. G. M. Merriman: *On a general theorem regarding divergent series, and its application to the double Fourier series.*

In the present paper a general theorem is proved which gives sufficient conditions that a double series, known to be summable ($C1$), should also be convergent. By a combination of this theorem with that concerning the summability of the double Fourier series, certain sufficient conditions for the convergence of the double Fourier series are obtained; these conditions are analogous to Dini's conditions in the case of the ordinary Fourier series, and were first introduced by Miss Hilda Geiringer (MONATSHEFTE FÜR MATHEMATIK UND PHYSIK, vol. 29 (1918)). The methods employed in proving the above results are generalizations of those used by Mr. S. Pollard in carrying through a similar investigation in the case of the ordinary Fourier series (PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 15 (1916)).

4. Professor C. N. Moore: *On necessary and sufficient conditions for convergence factors in double series.*

Sufficient conditions that a set of factors introduced into the terms of a double series summable by Cesàro's method should cause the resulting double series to be convergent have been given by the author in a paper that appeared in the TRANSACTIONS OF THIS SOCIETY in 1913. In the present paper some of these conditions are modified in such a manner that the resulting set has the property of being both necessary and sufficient. Furthermore, it is shown that if certain additional restrictions are made, we obtain the necessary and sufficient conditions that the function defined by the series with the convergence factors should approach the value to which the original series is summable when the convergence factors approach unity.

5. Professor C. N. Reynolds, Jr.: *Note on the map coloring problem.*

In this note it is shown that in Dr. Franklin's recent solution of the map coloring problem for any map of n regions where $n < N$ and $26 \leq N \leq 42$ (AMERICAN JOURNAL OF MATHEMATICS, vol. 44 (1922)), the N is subject to the narrower restriction $26 \leq N \leq 32$.

6. Professor R. L. Moore: *Concerning the prime parts of a continuum which separates its plane.*

In this paper it is shown that if, in a plane S, M is a bounded continuum which has more than one prime part and no prime part of M separates S, then in order that $S-M$ should be the sum of two mutually exclusive connected domains having M as their common boundary it is necessary and sufficient that the set G whose elements are the prime parts of M should be a simple closed curve of prime parts in the sense that it is disconnected by the omission of any two of its elements which are not identical. For a definition of prime part ("Primteil") of a continuum, see Hans Hahn, WIENER SITZUNGSBERICHTE, vol. 130 (1921).

7. Dr. W. Kraupner: *Integral solutions of the diophantine equations* $\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2 = \eta_1^2 + \eta_2^2 + \cdots + \eta_n^2$ *in the quadratic realm of rationality.*

This paper contains a generalization of the results ob-

tained by Hancock in his paper in Liouville's JOURNAL, (8), vol. 4 (1921), p. 327. It is shown that there exist an infinite number of integral solutions of the quadratic diophantine equations $\xi_1^2+\xi_2^2+\cdots+\xi_n^2=\eta_1^2+\eta_2^2+\cdots+\eta_n^2$. The investigation is based upon the irreducible quadratic equation $x^2-2a_1x-a_2=0$, a_1 and a_2 being rational integers, which defines an algebraic integer in the quadratic realm of rationality.

8. Professor Harris Hancock: *On algebraic equations whose roots are trigonometric functions.*

This paper considers algebraic equations whose roots are trigonometric functions, their discriminants, realms of rationality etc., and a method of derivation of certain reciprocal equations.

9. Professor E. W. Chittenden: *Properties of abstract sets implied by properties of the class of all continuous functions.*

Let P be an abstract class and $\delta(p,q)$ a non-negative function such that $\delta(p,p)=0$. The late Professor A. D. Pitcher studied the class C of all continuous real functions of a variable p with range P and obtained among other results the conditions implying that the space P be a compact, connected, metric space (A. D. Pitcher and E. W. Chittenden, *On the foundations of the calcul fonctionnel of Fréchet*, TRANSACTIONS OF THIS SOCIETY, vol. 19 (1918)).

The present paper proposes to extend and complete the results of Professor Pitcher. It considers the class C in the more general system (P, K) where K is a relation defining the points of accumulation of the subsets of P. The paper contains a set of independent properties of C, a characterization of those spaces P which admit a non-constant continuous function, and certain generalizations and modifications of previous results. It is found that with respect to the theory of continuous functions the notions point of accumulation and neighborhood (voisinage) are equivalent.

10. Professor A. D. Campbell: *The classification of linear families of conics in various domains.*

The domains considered in this paper are the ordinary complex domain, the real domain, and the Galois fields of order p^n $(p>2)$. The corresponding planes considered are

the complex projective plane, the real projective plane, and the Galois field projective plane. The classes of conics and of pencils of conics in these planes are well known; as also the nets and three- and four-parameter linear families of conics in the complex domain, and nets in the above Galois fields.

This paper deals with the nets and three- and four-parameter families in the real domain, also the three- and four-parameter families in the Galois fields. A geometrical description of all these classes is given by means of apolarity. The classes of families in the other domains are obtained as sub-classes of those in the complex domain or by interpretation therefrom. The classes of three- and four-parameter families are obtained by apolarity from the classes of pencils and the types of conics respectively. The classes of nets of conics are obtained by use of the associated cubic curves in the plane of the parameters of the nets.

11. Professor Mary E. Sinclair: *The isoperimetric problem with variable end points.*

Using the derivatives of the extremal-integral, necessary conditions are found for a minimum in the usual isoperimetric problem with variable end points. These conditions, strengthened in the usual manner, are found to be sufficient for a weak minimum. Extension to the isoperimetric problem is made of the Bliss geometric interpretation of critical value of radius of curvature. If r_1, r_2, C_1 and C_2 are radii and centers of curvature of the fixed curves at the two end points, P_1 and P_2, there exists for every value of r_1 a critical value $\bar{r}_2$ of r_2, increasing with r_1, projective to r_1, and such that, according as P_1P_2 and $P_2\bar{C}_2$ have the same or opposite direction, C_2 must or must not lie on $P_2\bar{C}_2$.

Our conditions may be expressed in a form very close to those of Merrill for his special type of problem. Again, as with Dresden's results in the non-isoperimetric case, they can be expressed in terms of particular solutions of the differential equation, $\Psi(w) + \mu V = 0$, and they reduce to Dresden's if the isoperimetric condition is dropped.

12. Professor Mary E. Sinclair: *The hanging chain with end points variable on curves in a plane.*

The conditions for a minimum in the isoperimetric problem with variable end points are applied to the problem of the

hanging chain of given length and uniform density. In the cases of one and of two variable end points a geometrical construction is given for the critical value, $\bar{r}_2$, of the radius of curvature.

13. Professor C. C. MacDuffee: *Covariants of differential forms of arbitrary order and degree.*

Covariants of differential forms of arbitrary order and degree have been studied by E. Pascal, ATTI DEI LINCEI, MEMORIE FISICHE, vol. 8 (1910). The definition of differential form used in this paper is essentially equivalent to that given by Pascal, but the subsequent treatment is quite different. A differential form of order h and degree r is defined as a polynomial of the type

$$\sum a_{i_1 i_2 \cdots i_m j_1 j_2 \cdots j_m} d^{i_1} x_{j_1} d^{i_2} x_{j_2} \cdots d^{i_m} x_{j_m},$$

the sum extending over all distinct partitions of r into positive summands $i_1, i_2, \ldots, i_m$, each $\leq h$, and $j_1, j_2, \ldots, j_m$ being chosen with repetitions allowed from among the integers $1, 2, \ldots, n$, so as to yield all possible distinct terms. The coefficients a are functions of $x_1, x_2, \ldots, x_n$. The principal theorem obtained, an analogue of Hilbert's theorem, is that every relative covariant of weight ϵ in at most n distinct differential operators can be expressed as a function with constant coefficients of certain elementary covariants and their differentials multiplied by the ϵth power of the identical covariant $| d_j x_i |$. These elementary covariants are roughly analogous to the polars of the ground forms in the algebraic theory.

14. Professor G. A. Miller: *Number of cycles of the same order in any substitution group.*

This paper appears in the present number of this BULLETIN.

15. Professor I. A. Barnett: *Note on linear differential equations with constant coefficients.*

This note considers linear differential equations in general analysis whose coefficients are constant with respect to the independent variable. It is shown that in case the coefficient system is either symmetric or skew-symmetric the constituents of the fundamental set of solutions are all of the form of exponentials and not products of exponentials by powers of the independent variable. The result for the

symmetric case is given without proof in a course of lectures of Professor Bôcher, and the statement is made that Weierstrass completely solved this case in 1858, ten years before he introduced elementary divisors.

16. Professor H. W. March: *Deflection of a rectangular plate clamped at its edges.*

The solution of the problem of the deflection of a rectangular plate clamped at its edges and under a given load has not previously been expressed in terms of functions which satisfy either the prescribed non-homogeneous differential equation or the corresponding homogeneous equation. W. Ritz (Crelle's JOURNAL, 1909) gave a solution in terms of a series of functions which satisfy the boundary conditions but which do not individually satisfy either differential equation.

The classical method of treating such problems has failed in this case because it has not been found possible to combine known solutions of the homogeneous equation with a solution of the non-homogeneous equation in such a way as to satisfy all of the boundary conditions. In this paper an apparently new type of solution of the homogeneous equation is given and the problem in question is solved for the case of a uniform load. It appears that the method can be readily extended to more general cases. The present method gives a result that is much better adapted than that of Ritz to numerical computation and to the discussion of the elastic surface.

17. Dr. Norbert Wiener: *A generalization of the Dirichlet problem.*

It is shown that the Dirichlet problem is a particular case of a problem solvable for any bounded open region and any continuous boundary conditions.

18. Professor Arnold Dresden: *On Brouwer's contributions to the foundations of mathematics.*

This paper has appeared in the January-February number of this BULLETIN.

19. Professor G. E. Wahlin: *On the application of the theory of ideals to diophantine equations.*

This paper has appeared in the March-April number of this BULLETIN.

20. Professor Florian Cajori: *J. H. Rahn's mathematical symbols.*

The symbols in Rahn's Algebra written in German (1659) are compared with those found in the English translation (1668). There are set forth certain incorrect statements due to John Collins and John Wallis regarding John Pell's rôle in the preparation of the original edition as well as of the translation.

21. Professor Dunham Jackson: *A generalized problem in weighted approximation.*

In a recent note in this BULLETIN (June, 1923), the author has discussed the convergence of the trigonometric sums $T_n(x)$ determined as approximations to a given continuous function $f(x)$ by the condition that the integral of the square of the error, multiplied by a weight-function $\varrho(x)$, shall be a minimum. The function $\varrho(x)$ was assumed to be bounded, and to have a positive lower bound. The purpose of the present paper is to deal with the somewhat less simple problem of convergence that arises if the greatest lower bound of $\varrho(x)$ is zero. There is occasion incidentally to develop the theory of the existence and uniqueness of discontinuous approximating functions of a high degree of generality. The discussion is put in such form as to be applicable when the square of the error is replaced by the mth power of its absolute value.

22. Professor E. B. Van Vleck: *On the zeros of polynomials.*

In a very interesting memoir in the ANNALES DE L'ÉCOLE NORMALE SUPÉRIEURE, 1923, Montel shows, among other results, that, when the first $r+1$ coefficients $1, a_1, \ldots, a_r$ of the equation

$$1 + a_1x + a_2x^2 + \cdots + a_rx_r + a_{r+1}x^{r+r_1} + \cdots + a_{r+n}x^{r+r_n} = 0 \quad (r_i \geqq i)$$

are given, the r roots of smallest absolute value lie within a circle whose radius depends only upon r and the number of terms $n+r+1$ in the equation, and he determines this radius for $r=1$ and $a_1=1$, and for $r=2$, $a_1=0$. In this paper these results are supplemented by the determination of such a radius for any value of r when each $r_i=i$. It is also ascertained under what conditions an arbitrarily selected set of r coefficients a_i will limit in absolute value the r smallest roots and their upper limit is then determined when the degree of the equation is given.

23. Professor Arnold Emch: *On the Weddle surface and analogous loci.*

This paper starts from the geometric definition of the Weddle surface as the locus of the vertices of all quadric cones through six points in space. The locus of the vertices of quadric cones through seven points is a sextic S of genus 3. All quadrics through the seven points pass through an eighth fixed point; hence the sextic S is also the locus of vertices of quadric cones through the 8 base points of the net of quadrics. S cuts each of the 28 lines, connecting the base points, in two points. There are 28 Weddle surfaces through S, each having 6 of the 8 base points as nodes. The problem is also investigated for the cubic and Geiser transformations connected with the 8 base points.

The locus of the vertices of cubic cones through 10 points B is found to be a surface of order 15 with the B's as cubic nodes, passing through each of the 45 lines joining the B's. The locus of the vertices of cubic cones through 11 points is a curve of order 189 cutting each of the 45 lines in 12 points; its genus is determined. Finally the paper gives the loci of the vertices of $n$ic cones through $(n(n+3)/2)+1$ and through $(n(n+3)/2)+2$ points in space.

24. Professor Tomlinson Fort: *Note on Dirichlet and factorial series.*

In a note published in the TRANSACTIONS OF THIS SOCIETY, January, 1922, the author studied a class of series which included ordinary Dirichlet series and factorial series as special cases. The present paper extends the results of that note.

25. Dr. R. L. Wilder: *On the dispersion sets of a connected point set.*

A proper subset H of a point set M is called a dispersion set of M if $M-H$ is totally disconnected. H is called a primitive dispersion set of M if no proper subset L of H exists such that $M-L$ is totally disconnected. A special study is made of the properties of connected sets that contain primitive dispersion sets. If H is a primitive dispersion set of a connected set M consisting of a finite number of points (the existence of connected sets with primitive dispersion sets consisting of a finite number of points

has recently been established by Knaster and Kuratowski, (FUNDAMENTA MATHEMATICAE, vol. 2 (1921), pp. 206–255), it is shown that M is the sum of a finite number of mutually exclusive connected sets, each of which consists of more than one point and contains one and only one point of H. It is also shown that if a connected set M contains a primitive dispersion set H which consists of a finite number of points, then H is the only primitive dispersion set of M.

26. Mr. J. A. Shohat: *The theory of closure of Tchebychef's polynomials for an infinite interval.*

If $\varphi_n(x)$ $(n = 0, 1, 2, \ldots)$ denote a system of Tchebychef's orthogonal and normal polynomials corresponding to the interval (a, b) with the characteristic function $p(x)$ not negative in (a, b), then $\int_a^b p(x)\Phi_m(x)\Phi_n(x)dx$ is equal to unity if m and n are equal, and to zero if m and n are unequal. In the case of a finite interval (a, b), every function $f(x)$ which with its square is integrable in (a, b) satisfies an important *closure equation.*

In this paper, the author investigates the closure equation for an infinite interval (a, ∞), and shows that the closure equation holds if, for x sufficiently large, $p(x) < e^{-|x|^\lambda}$, with $\lambda > 1/2$ for $a = 0$, and $\lambda > 1$ for $a = -\infty$. Particular cases are the polynomials of Hermite-Tchebychef and those of Laguerre-Tchebychef. The closure equation is not true in general. A general example is given of a system of Tchebychef's polynomials in $(0, \infty)$ for which the closure equation is not true.

27. Professor C. H. Sisam: *On curves whose first polars contain a pencil of lines.*

The algebraic plane curves for which every right line through a fixed point is a component of a first polar with respect to the curve are completely determined.

28. Professor H. T. Davis: *Integral equations as differential equations of infinite order.*

In this paper an equivalence is pointed out between linear differential equations of infinite order and integral equations of either the Volterra or Fredholm type which have solutions analytic within the interval of integration. An existence theorem for the differential equation of infinite order with constant coefficients in both the homogeneous and non-

homogeneous cases is given and the solutions are shown to satisfy a singular integral equation of Volterra type.

29. Professor L. E. Dickson: *On the theory of numbers and generalized quaternions.*

This paper opens with a new simple proof by the method of descent of Bachet's theorem that every positive integer is a sum of four integral squares, and the new generalization that the roots of the four squares may be chosen to satisfy four assigned linear congruences.

Next there is given a simpler proof of the author's formulas for all integral solutions of $x^2+y^2+z^2+w^2=uv$. These formulas and generalized quaternions occur in the author's *Algebras and Their Arithmetics*, University of Chicago Press, 1923, pp. 187–198. The present paper discusses an arithmetic of generalized quaternions for which there exists a greatest common left divisor, although it cannot be found by a process of division with remainders of decreasing norms. The new method is first applied to the arithmetic of quaternions and shows the difference between the integral quaternions of Lipschitz and those of A. Hurwitz. Application is made to diophantine equations. The memoir will appear in the AMERICAN JOURNAL.

30. Professor L. E. Dickson: *Quadratic fields in which factorization is always unique.*

This paper appears in this number of this BULLETIN.

31. Professor A. B. Coble: *Geometric interpretation of the expression of an algebraic form as a determinant.*

Dickson and Everett have discussed the possibility of expressing a given algebraic form in k variables as a determinant whose elements are forms of the same order. It is the purpose of this paper to show that this possibility rests when $k=3$ upon the existence of coresidual point groups on the given algebraic curve; when $k=4$ upon the existence of coresidual linear systems of curves on the given algebraic surface; etc. If the determinant is to be symmetric, contact systems come into play.

32. Mr. H. S. Vandiver: *Sets of three consecutive integers which are quadratic or cubic residues of primes.*

In this paper the author proves several theorems such as the following: If p is of one of the forms $40k+1$,

$40k+11$, $40k+29$, $40k+39$, then $a+1$, a, $a-1$ are all quadratic residues of p, where $a^2 \equiv 5 \pmod{p}$ and 5 is a biquadratic residue of p.

33. Mr. H. S. Vandiver: *On Kummer's memoir of* 1857 *concerning Fermat's last theorem.*

In two previous papers under this title the author pointed out that Kummer made some errors in the proofs of certain theorems. In the first paper (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6 (1920), pp. 266–269) these theorems were numbered I to IV. The second paper (this BULLETIN, vol. 28 (1922), pp. 400–407) contained proofs of Theorems I and IV. The present paper is devoted to the proof of Theorem III and an extension of Theorem II is also proved which enables the author to establish the result: If the class numbers of the field $\Omega(\alpha)$, $\alpha = e^{2i\pi/p}$, is divisible by p but not by p^2, then $x^p+y^p+z^p = 0$ is not satisfied in integers in the field $\Omega(\alpha+\alpha^{-1})$, not zero, p being an odd prime. This includes Kummer's results as special cases.

34. Professor H. L. Smith: *Necessary and sufficient conditions for the existence of a class of Stieltjes integrals.*

In this paper necessary conditions are obtained for the existence of the Stieltjes integral $\int_0^1 \psi(t)d\varphi(t)$ which are also sufficient if the associated curve $x=\varphi(t)$, $y=\psi(t)$ $(0\leqq t\leqq 1)$ is continuous and such that its multiple points are a reducible set. The conditions remain sufficient for discontinuous curves which can be embedded in continuous curves of a certain character. A corollary is that every simple closed (continuous) curve is squarable if the line integral $\int ydx$ exists over it. An example is given of a simple closed squarable curve over which the line integral $\int ydx$ does not exist.

35. Professor S. Lefschetz: *On polyhedra in euclidean n-space.*

In this paper are first developed certain properties of covering manifolds, which are then applied to the proof of this theorem: A bounded, non-singular, $(n-1)$-dimensional polyhedron in euclidean n-space bounds a finite region which together with the polyhedron is homeomorphic to an n-cell plus its boundary.

36. Professor R. L. Moore: *An extension of the theorem that no perfect set is countable.*

Sierpinski has shown that if the closed and bounded point set M is the sum of a countable number of mutually exclusive closed point sets $M_1, M_2, M_3, \ldots$ then M is not connected. In the present paper it is shown that, if the additional requirement is imposed that the point sets $M_1, M_2, M_3, \ldots$ should be connected, then not every one of them contains a limit point of the sum of all the others. It is shown that this latter conclusion does not hold if either this additional requirement or the requirement that M be bounded be omitted.

37. Professor Louis Brand: *Note on the integral theorems of vector analysis.*

The integral theorems of vector analysis fall into two principal groups. In the first group the volume integral of $\nabla\omega$ where ω denotes a scalar, vector, or dyadic point function is transformed into an integral of $\boldsymbol{u}\omega$ over the bounding surface, $\boldsymbol{u}$ being the unit surface normal. In the present note it is shown that the second class of theorems, which transform surface to circuit integrals, may be deduced from the first class. The proof hinges on the fact that $\boldsymbol{u}$ may be regarded as the gradient of a scalar point function whose level surfaces are parallel to the original surface.

38. Miss I. M. Schottenfels: *The Kürschak field of complex numbers.*

The author extends to the complex number field the results obtained by J. Kürschak in his paper in Crelle's JOURNAL, vol. 142 (1913). The work of Ostrowski (ACTA MATHEMATICA, vol. 41 (1918)) is also reorganized and extended.

39. Miss I. M. Schottenfels: *The error in Hartog's proof of the Zermelo theorem.*

The author points out an incompleteness in Hartog's proof of Zermelo's theorem which appeared in MATHEMATISCHE ANNALEN, vol. 76 (1915) and furnishes a complete proof, far simpler and more concise than those published by Kuratowski in volume 2 of the FUNDAMENTA MATHEMATICAE.

ARNOLD DRESDEN,
Assistant Secretary.

A CHARACTERIZATION OF SURFACES OF TRANSLATION*

BY E. P. LANE

Let the non-homogeneous cartesian coordinates x, y, z of an arbitrary point on a surface be given as analytic functions of two independent variables u, v by equations of the form

$$x = U_1 + V_1, \quad y = U_2 + V_2, \quad z = U_3 + V_3,$$

where U_1, U_2, U_3 are functions of u alone and V_1, V_2, V_3 are functions of v alone. Such a surface is called a surface of translation because it can be regarded in two ways as generated by the motion of a curve which is translated so that its various points describe congruent curves. It is the purpose of this note to give another characterization of surfaces of translation, based on some notions which have arisen in the study of projective differential geometry.

Since, at each surface point P, the tangents to the curves $u = \text{const.}$ and $v = \text{const.}$ separate the two asymptotic tangents harmonically,† it follows that the parametric net is a conjugate net. Therefore the tangents to the curves $u = \text{const.}$, constructed at the points of a fixed curve $v = \text{const.}$, form a developable surface. The point where the tangent to the curve $u = \text{const.}$ through P touches the edge of regression of this developable is one of the two *ray points* of P, with respect to the parametric conjugate net. The other ray point is similarly defined on the other parametric tangent. The line joining these two ray points is called the *ray* of P. The totality of rays of all the points on the surface constitutes a congruence called the *ray congruence* of the fundamental conjugate net. And the

* Presented to the Society, April 25, 1924.

† Lie, MATHEMATISCHE ANNALEN, vol. 14 (1879), pp. 332-337.

curves on the surface which correspond to the developables of the ray congruence are the *ray curves*. These definitions serve for any conjugate net on an arbitrary surface, and are due to Wilczynski.

But since our surface is a surface of translation and the parametric conjugate net is the net of generating curves, it follows that the developable circumscribing the surface along a curve $u = \text{const.}$, or a curve $v = \text{const.}$, is a cylinder. Therefore the ray points of every surface point are both at infinity, and the ray of every surface point is entirely at infinity. The ray congruence consists therefore of lines all lying in one plane, namely, the plane at infinity, and the ray curves are indeterminate.

One of my students, Mr. M. L. MacQueen, in his master's thesis, University of Wisconsin, 1923, has studied the class of surfaces on each of which there exists a conjugate net with indeterminate ray curves. He shows that the rays of such a conjugate net all lie in one plane, which we may call the *ray plane* of the net. He shows, furthermore, that every surface on which there exists a conjugate net with indeterminate ray curves can be projected into a surface of translation by precisely the projective transformation which carries the ray plane into the plane at infinity.

We may therefore characterize surfaces of translation by saying that *surfaces of translation are the only surfaces that have the property that on each of them there exists a conjugate net with indeterminate ray curves and with ray plane at infinity.*

The University of Chicago

CONCERNING A SUGGESTED AND DISCARDED GENERALIZATION OF THE WEIERSTRASS FACTORIZATION THEOREM*

BY L. L. DINES

1. *Introduction.* A basic theorem in the theory of analytic implicit functions, proven by Weierstrass, may for our purpose be stated as follows:

Let (1°), $f(y; x_1, \ldots, x_n)$ *be analytic at the origin and vanish there; and* (2°), $f(y; 0, \ldots, 0)$ *be not identically zero. Then, throughout a certain neighborhood of the origin, there holds an identity of the form*†

$$(1) \qquad f(y; x_1, \ldots, x_n) = (P_0 y^m + P_1 y^{m-1} + \cdots + P_m)\, g(y; x_1, \ldots, x_n)$$

where $g(y; x_1, \ldots, x_n)$ *is analytic and does not vanish at the origin; and where* P_j, $j = 0, 1, \ldots, m$, *is an analytic function of* $x_1, x_2, \ldots, x_n$ *and for* $j > 0$ *vanishes when* $x_1 = x_2 = \cdots = x_n = 0$.

In his *Madison Colloquium Lectures,* Osgood called attention to the fact that the hypothesis (2°) may be omitted in the case of a function $f(y; x)$ of only two variables, without disturbing the validity of the conclusion; and suggested tentatively but without proof that the theorem in this stronger form might be true for a function $f(y; x_1 \ldots, x_n)$ of $n+1$ variables. In a later paper‡ he showed very definitely that the theorem is *not* true in general, with the omission of the hypothesis (2°). His proof of this fact consisted in the exhibition of a function of the form

$$(2) \qquad f(y; x_1, x_2) \equiv x_1 - x_2 F(y)$$

which is not factorable in the form (1).

---

* Presented to the Society, September 7, 1923.

† The theorem of Weierstrass states, in addition, that $P_0 = 1$, and that m is equal to the degree of the term of lowest degree in the series $f(y; 0, \ldots, 0)$.

‡ TRANSACTIONS OF THIS SOCIETY, vol. 17, page 4.

The object of the present note is to consider further the possibility of an identity of form (1) in case (2°) is not satisfied. In the next article, we shall derive a simple necessary condition (Theorem I) for the existence of such an identity, and we shall show in particular (Corollary II) that a function of the form (2) is not factorable in the manner under consideration unless $F(y)$ is a *rational* function of y.

2. *A Necessary Condition.* If an identity of form (1) holds, then there is determined a second identity

$$\Phi \cdot f = P \tag{3}$$

where $\Phi = 1/g$, and $P = P_0 y^m + P_1 y^{m-1} + \cdots + P_m$. Let us suppose the power series expansions of the functions Φ, f, and P to be written in the following forms:

$$\begin{aligned} \Phi(y; x_1, \ldots, x_n) &= \Phi^{(0)} + \Phi^{(1)} + \Phi^{(2)} + \cdots \\ f(y; x_1, \ldots, x_n) &= f^{(k)} + f^{(k+1)} + f^{(k+2)} + \cdots \\ P(y; x_1, \ldots, x_n) &= P^{(k)} + P^{(k+1)} + P^{(k+2)} + \cdots \end{aligned} \tag{4}$$

where $\Phi^{(i)}$, $f^{(i)}$, and $P^{(i)}$ are homogeneous polynomials of degree i in $x_1, \ldots, x_n$. The polynomials $\Phi^{(i)}$ and $f^{(i)}$ have coefficients which are power series in y, while the coefficients of $P^{(i)}$ contain y to no higher power than the m th. Since g is analytic and does not vanish at the origin, it follows that $\Phi^{(0)}$ has a constant term different from zero. From the identity (3) it follows that f and P must begin with terms of the same degree in $x_1, \ldots, x_n$, and since we are considering the case in which (2) is not satisfied, i. e., in which $f(y; 0, \ldots, 0) \equiv 0$, we shall assume that this degree k is greater than zero.

Substituting the series (4) in (3), and equating polynomials of lowest degree on the two sides we obtain the identity

$$\Phi^{(0)} f^{(k)} = P^{(k)}. \tag{5}$$

A homogeneous polynomial of degree k contains power products of the form $x_1^{j_1} x_2^{j_2} \cdots x_n^{j_n}$ where $j_1 + j_2 + \cdots + j_n = k$. Let the sequence of all such power products in any definite order be denoted by $X_1, X_2, \ldots, X_N$. Then we have

$$\begin{aligned} f^{(k)} &= f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N, \\ P^{(k)} &= P_1^{(k)} X_1 + P_2^{(k)} X_2 + \cdots + P_N^{(k)} X_N, \end{aligned}$$

where $f_i^{(k)}$ is, for every i, a power series in y, while $P_i^{(k)}$ is, for every i, a polynomial in y. Substituting these expressions for $f^{(k)}$ and $P^{(k)}$ in (5) and equating coefficients of like power products, we find that

$$\Phi^{(0)} f_i^{(k)} = P_i^{(k)}, \quad i = 1, 2, \ldots, N.$$

Therefore

$$\Phi^{(0)} = \frac{P_1^{(k)}}{f_1^{(k)}} = \frac{P_2^{(k)}}{f_2^{(k)}} = \cdots = \frac{P_n^{(k)}}{f_N^{(k)}}; \tag{6}$$

or otherwise expressed, for every i and j for which $f_i^{(k)}$ and $f_j^{(k)}$ are different from zero,

$$\frac{f_i^{(k)}}{f_j^{(k)}} = \frac{P_i^{(k)}}{P_j^{(k)}} = \textit{a rational function of } y.$$

Hence we may state the following theorem.

THEOREM I. *If the leading homogeneous polynomial in $x_1, x_2, \ldots, x_n$ of $f(y; x_1, \ldots, x_n)$ be*

$$f^{(k)} = f_1^{(k)} X_1 + f_2^{(k)} X_2 + \cdots + f_N^{(k)} X_N$$

where $X_1, X_2, \ldots, X_N$ are the various power products of degree k in $x_1, x_2, \ldots, x_n$ and $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ are power series in y, then a necessary condition that f admit factorization in the form (1) *is that the ratio formed from each pair of non-vanishing coefficients $f_i^{(k)}, f_j^{(k)}$ be a rational function of y.*

COROLLARY I. *If any one of the sequence of coefficients $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$ is a rational function, then all of these coefficients must be rational if $f(y; x_1, \ldots, x_n)$ is to admit factorization in form* (1).

COROLLARY II. *A necessary condition that the function*

$$f(y; x_1, x_2) \equiv x_1 - x_2 F(y)$$

admit factorization in the form (1) *is that $F(y)$ be a rational function of y.*

3. *Sufficient Conditions.* If the function $f(y; x_1, \ldots, x_n)$ is homogeneous in $x_1, \ldots, x_n$, that is if $f = f^{(k)}$, the condition of our theorem is *sufficient*. For if there exist poly-

nomials in y, viz. $P_1^{(k)}, P_2^{(k)}, \ldots, P_N^{(k)}$ such that $f_1^{(k)} : f_2^{(k)} : \cdots : f_N^{(k)} = P_1^{(k)} : P_2^{(k)} : \cdots : P_N^{(k)}$, there will exist such polynomials in which the lowest powers of y are respectively equal to the lowest powers of y in the series $f_1^{(k)}, f_2^{(k)}, \ldots, f_N^{(k)}$. Hence there will exist a series $\Phi^{(0)}(y)$ with non-vanishing constant term satisfying (6). It is then easy to construct the polynomial $P^{(k)}$ satisfying the relation $\Phi^{(0)} f^{(k)} = P^{(k)}$. In view of the homogeneity of f this relation may be written in the form $\Phi f = P$. From this may be obtained the equivalent form (1).

If $f(y; x_1, \ldots, x_n)$ is not homogeneous in $x_1, x_2, \ldots, x_n$ the condition of the theorem is *not sufficient.*

THE UNIVERSITY OF SASKATCHEWAN

THE CLASS NUMBER RELATIONS IMPLICIT IN THE DISQUISITIONES ARITHMETICAE*

BY E. T. BELL

1. *Introduction.* The point of this note is its moral, which is to the effect that in arithmetic attention to trifles sometimes leads to beautiful and recondite truths. In particular, certain important expansions of Kronecker and Hermite relating to the number $F(n)$ of uneven classes of binary quadratic forms of negative determinant $-n$ are implicit in § 292 of the *Disquisitiones Arithmeticae* of Gauss, and might have been read off from there at a glance by anyone familiar with the *Fundamenta Nova* of Jacobi, thirty years before Kronecker first came upon them by the devious route of complex multiplication. The relevant trifle in this instance is changing the sign of an arbitrary constant throughout an algebraic identity.

* Presented to the Society, April 5, 1924.

It is well known that the tenth of Kronecker's relations* is equivalent to his expansion

$$(\text{XI}) \quad 12\sum_{0}^{\infty} E(n)\,q^n = \left[\sum_{-\infty}^{\infty} q^{n^2}\right]^3 = \vartheta_3^3(q),\ E(n) = F(n) - F_1(n),$$

where $F_1(n)$ is the number of even classes, which is merely the algebraic expression of the theorem of Gauss (loc. cit.) on the number of representations of n as a sum of three squares; and it has recently been remarked by Mordell† that Hermite's important formula

$$(1) \quad \sum_{0}^{\infty} F(8n+3)\,q^{8n+3} = (q + q^9 + q^{25} + \cdots)^3 = \tfrac{1}{8}\vartheta_2^3(q^4),$$

and those of Kronecker (changed from k, K to ϑ notation),

$$(2) \quad 4\sum_{0}^{\infty} F(4n+2)\,q^{4n+2} = \vartheta_3(q^4)\,\vartheta_2^2(q^4),$$

$$(3) \quad 4\sum_{0}^{\infty} F(4n+1)\,q^{4n+1} = \vartheta_2(q^4)\,\vartheta_3^2(q^4),$$

are implicit in (XI) and can be obtained from it by elementary algebra. Note in passing that (1) is simply the algebraic translation of § 293 of the *Disquisitiones Arithmeticae*. Both Kronecker and Hermite made much use of (1), (2), (3) in their successive simplification of the classical relations and in further deductions, and these identities reappear repeatedly in works of later writers. It is interesting therefore to see that (1), (2), (3) follow from (XI) even more simply than as shown by Mordell; in fact they result from (XI) by changing the sign of q. That is, (XI), (1), (2), (3) are immediate consequences of Gauss' theorem. The deduction of class number relations from these four identities by combining them with the expansions of Jacobi is not only classical but obvious, and it seems surprising that the immediate step of translating the application which Gauss made of his theorem, to representations as

* Cf. Dickson's *History*, vol. 3, pp. 108, 109, 113.

† MESSENGER OF MATHEMATICS, vol. 45 (1915) pp. 78, 79.

sums of four squares, into the equivalent class number relation was not taken much earlier than it was.

2. *Consequences of Gauss' Theorem.* In (XI) change q into $-q$:

$$(\mathrm{XI}_1) \qquad 12\sum_{0}^{\infty}(-1)^n E(n)q^n = \left[\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right]^3 = \vartheta_0^3(q).$$

Let $n_1, n_2, \ldots$ denote arbitrary integers $\gtreqless 0$, $m_1, m_2, \ldots$ odd integers $\gtrless 0$, and write $N(n = x^2+y^2+z^2)$ for the number of integer solutions $x, y, z \gtreqless 0$ of $n = x^2+y^2+z^2$. Equate coefficients of q^n in (XI_1):

$$\begin{aligned}
&12(-1)^n E(n)\\
&\quad = N(n = 4n_1^2+4n_2^2+4n_3^2)+N(n = 4n_1^2+m_2^2+m_3^2)\\
&\quad + N(n = m_1^2+4n_2^2+m_3^2)+N(n = m_1^2+m_2^2+4n_3^2)\\
&\quad - N(n = m_1^2+m_2^2+m_3^2)-N(n = m_1^2+4n_2^2+4n_3^2)\\
&\quad - N(n = 4n_1^2+m_2^2+4n_3^2)-N(n = 4n_1^2+4n_2^2+m_3^2),\\
&\quad = N(n = 4n_1^2+4n_2^2+4n_3^2)+3N(n = 4n_1^2+m_2^2+m_3^2)\\
&\quad - N(n = m_1^2+m_2^2+m_3^2)-3N(n = m_1^2+4n_2^2+4n_3^2).
\end{aligned}$$

Since

$$\begin{aligned}
4n_1^2+4n_2^2+4n_3^2 &\equiv 0 \mod 4,\\
4n_1^2+m_2^2+m_3^2 &\equiv 2 \mod 4,\\
m_1^2+m_2^2+m_3^2 &\equiv 3 \mod 8,\\
m_1^2+4n_2^2+4n_3^2 &\equiv 1 \mod 4,
\end{aligned}$$

we have $12\,E(n) = N(4n = 4n_1^2+4n_2^2+4n_3^2)$, which is merely (XI) again, and

$$\begin{aligned}
N(8n+3 = m_1^2+m_2^2+m_3^2) &= 12\,E(8n+3),\\
N(4n+2 = 4n_1^2+m_2^2+m_3^2) &= 4\,E(4n+2),\\
N(4n+1 = m_1^2+4n_2^2+4n_3^2) &= 4\,E(4n+1);
\end{aligned}$$

which are equivalent respectively to

$$\begin{aligned}
(\sum q^{m_1^2})^3 &= 12\sum q^{8n+3}E(8n+3),\\
\sum q^{4n_1^2}\times(\sum q^{m_1^2})^2 &= 4\sum q^{4n+2}E(4n+2),\\
\sum q^{m_1^2}\times(\sum q^{4n_1^2})^2 &= 4\sum q^{4n+1}E(4n+1).
\end{aligned}$$

But these are identical respectively with (1), (2), (3), since

$$E(8n+3) = \tfrac{8}{3}F(8n+3),\ E(4n+2) = F(4n+2),\ E(4n+1) = F(4n+1).$$

THE UNIVERSITY OF WASHINGTON

NUMBER OF CYCLES OF THE SAME ORDER IN ANY GIVEN SUBSTITUTION GROUP*

BY G. A. MILLER

1. *Introduction.* If G is a transitive substitution group and if the subgroup composed of all the substitutions of G which omit a given letter is of degree $n-\alpha$, then there are exactly α substitutions involving no letters except possibly those of G which are commutative with every substitution of G. These α substitutions include the identity. If $\alpha > 1$ the remaining $\alpha-1$ substitutions may or may not appear in G. From this well known theorem, it follows directly that G involves exactly $\alpha-1$ sets of conjugate cycles which are such that no two distinct cycles of the set have a common letter. Each of these cycles appears in g/n different substitutions of G, where g denotes the order of G. A necessary and sufficient condition that a transitive substitution group be regular is that no two of its sets of conjugate cycles have a common letter.

When no two conjugate cycles of G have a letter in common it is evident that every pair of cycles in a set of conjugates must be commutative, but these cycles may also be commutative when G is non-regular. When G involves at least one set of conjugate cycles which has the property that every pair of cycles in the set is composed of commutative cycles, G must be imprimitive unless all these cycles involve the same letters and are also of prime order. In this special case, G is evidently always primitive. From the fact that a cycle of prime degree p is transformed into each of its various powers which are incongruent to 1 (mod p) only by substitutions of degree $p-1$ on the letters of this cycle it results directly that such substitutions involve cycles

* Presented to the Society, December 29, 1923.

that are not commutative for every such power. Hence we have the following theorem.

THEOREM I. *If every set of conjugate cycles of a transitive substitution group is composed of relatively commutative cycles, then the subgroups composed separately of all the substitutions of the group which omit a letter must omit a number of letters which is divisible by the product of the prime factors of the order of the group.*

In particular, it results from this theorem that a transitive group which has the property that all of its sets of conjugate cycles are composed of cycles that are relatively commutative must also involve at least one set of conjugate cycles that is composed of cycles such that no two of them have a common letter.

Suppose that G is intransitive and involves λ transitive constituents of orders $g_1, g_2, \ldots, g_\lambda$, respectively. Let k represent the order of an arbitary cycle of some substitution of G. From the fact that the number of the letters contained in every set of conjugate cycles of any substitution group is equal to the order of the group, whenever each cycle is counted for every substitution in which it appears, it results directly that the numbers of the cycles of order k in the transitive constituents of G are, respectively,

$$\frac{m_1 g_1}{k}, \quad \frac{m_2 g_2}{k}, \quad \ldots, \quad \frac{m_\lambda g_\lambda}{k},$$

where $m_1, m_2, \ldots, m_\lambda$ are positive integers or zero, and that the number of cycles of order k contained in G is mg/k, where g is the order of G and $m = m_1 + m_2 + \cdots + m_\lambda$. It should be noted that this theorem is independent of the way in which G is constructed by means of its transitive constituents, and that there are an infinite number of possible groups for every arbitrary pair of values of m and k, since all of the cycles of the same order contained in the symmetric group of degree n are conjugate under this group.

In particular, it results from the theorem noted above that if at least two of the transitive constituents of G in-

volve cycles of the same order, the number of letters found in all of the cycles of this order contained in G must always exceed the order of G, each cycle being counted for every substitution in which it appears. It is evident that the numbers $m_1, m_2, \ldots, m_\lambda$, cannot exceed the degrees of the corresponding transitive constituents diminished by one, and when they attain these maximal values k must be a prime number p and all the cycles of the corresponding constituents must be of order p. In particular, the order of such a constituent must be of the form p^m. It should also be noted that all the substitutions of a given order k may be conjugate under the group while the cycles of this order do not constitute a single set of conjugates, and that conversely all the cycles of order k contained in a group may constitute a single set of conjugates while the substitutions of this order do not have this property.

2. *Smallest Number of Cycles of Order p in a Substitution Group of Order p^m.* If G is a substitution group of order p^m, where p is a prime number, it results directly from the theorems noted above that the number of cycles of order p contained in G cannot be less than g/p, and this minimum can be attained only when G is transitive. It is also evident that this minimum cannot be attained unless $p = 2$, since all the cycles of order p must be conjugate under G when it is attained. Hence we shall assume in the rest of this section, unless the contrary is stated, that $p = 2$, and that the number of the transpositions contained in the substitutions of G is exactly $g/2$.

Since all of these transpositions are conjugate under G, they must appear in an invariant substitution of G. In fact, in every substitution group of order p^m, where p is any prime number, all the cycles of order p that appear in an invariant substitution of order p constitute a complete set of conjugates and must therefore appear in more than one substitution whenever the group is non-regular. In the particular case under consideration, G involves only

one set of conjugate cycles of order 2, and hence the substitutions of order 2 contained in G, if there is more than one such substitution, generate an abelian invariant sub-group.

Except in the trivial case when $g = 2$ there must be a substitution s of order 4 in G which permutes all the transpositions of the invariant substitution of order 2 contained in G, since the transitive group according to which these transpositions are transformed contains an invariant substitution of order 2. The group generated by s and the substitutions contained in G whose orders divide 2, is composed of substitutions of order 4 and of degree n, where n is the degree of G, in addition to the subgroup composed of the substitutions whose orders divide 2. Hence all of the substitutions of order 4 have a common square and the group generated by them is abelian.

If G involves no substitutions whose orders exceed 4, the cycles of the invariant substitution of order 2 contained in the transitive group G must be transformed under G according to a regular group, since this group is transitive and involves only substitutions of order 2 besides the identity. Hence all of the substitutions of order 4 must have a common square and G can involve only one substitution of order 2, as otherwise all of the subgroups corresponding to subgroups of order 2 in the regular group according to which the cycles of order 2 are permuted could not be abelian. The order of this regular group cannot exceed 4, since a substitution of order 4 must transform into its inverse every substitution of this order which is not generated by it. This proves the following theorem.

THEOREM II. *If a transitive substitution group of order p^m, where p is a prime number, involves no substitution of order p^2, and if the number of its cycles of order p is p^{m-1}, then it must be regular and simply isomorphic with one of the following three groups: the quaternion group, the cyclic group of order* 4, *or the group of order* 2.

When $m = n-2$ for the cycles of a given order k found in the transitive group G of degree n, m must be unity for

the cycles of the other possible order and the latter cycles must constitute a single set of conjugates under G. It was noted above that k is either 2 or 4 when the order of G is a power of a prime number. From the theorem at the end of § 1, it results that when $k=2$, G must be one of three groups. The fact that G must be the octic group when $k=4$ will be proved in the following section.

3. *Minimum Number of Cycles of Order* 4. When the cycles of order 4 in the transitive group G constitute a single set of conjugates, the subgroup composed of all the substitutions of G which omit a letter must involve at least one substitution besides the identity which omits at least two letters, since G cannot be regular and each of the cycles of the invariant substitution of order 2 contained in G is of order 2. It will first be proved that the cycles of order 2 found in the squares of the cycles of order 4 contained in G may be assumed to appear in the invariant substitution of order 2.

If this were not the case none of these squares could be commutative with all the cycles of this invariant substitution, and hence these cycles would be transformed according to cycles of order 4 by the cycles of order 4 contained in G. The transitive group according to which these cycles are transformed has a degree which is one-half the degree of G, and hence two cycles of order 4 in G would correspond to one cycle of this order in this transitive group. The cycles of order 4 in G would therefore be transformed under G according to an imprimitive group having two letters in each of these systems of imprimitivity. In the transitive group according to which these cycles would be transformed, there would therefore be again a single set of conjugate cycles of order 4, while each of the remaining cycles would be of order 2. Since this transitive group would have a lower order than the original group, we may assume for the time being that G is a transitive group of lowest order having the properties

in question, and hence the squares of its cycles of order 4 are found in the invariant substitution of order 2.

Since the substitutions of G transform the cycles of the invariant substitution of order 2 according to a group which contains only operators of order 2, this group must be abelian; and since it is transitive, it must also be regular. If it is of order 2, G must be the octic group. If its order exceeds 2, the cycles of order 4 can appear in only one of the co-sets of G which correspond to the various substitutions of order 2 according to which the cycles in the invariant substitutions are transformed under G, since all of these cycles of order 4 would have to appear in such a co-set if one such cycle appears there, and all the substitutions of these co-sets must involve all the letters found in G.

Therefore the order of the regular group in question cannot exceed 4, since at least one-fourth of the substitutions of G must have orders which exceed 2. Hence each substitution of order 4 contained in G, even when the squares of the cycles of order 4 contained in G are not found in the invariant substitution of order 2, is regular, and hence only one-fourth of the substitutions of G have an order greater than 2. It follows that G is either the octic group or the direct product of this group and an abelian group of type (1, 1, 1, . . .). Hence the degree of G cannot be less than half its order, and G must be the octic group if all its cycles of order 4 are conjugate.

4. *Non-Prime Power Transitive Groups in which* $m = n-2$. When the order of the transitive group G is not a power of a prime number and the number of the cycles of order k contained in G is $(n-2)g/k$, where n is the degree of G, it results directly that all the cycles of G are of prime order. When $n = 3$, G is the symmetric group of order 6; and when $n = 4$, G is the alternating group of order 12. We shall therefore assume in what follows that $n > 4$ unless the contrary is stated.

It will be convenient to use the following general theorem.

THEOREM III. *If a transitive substitution group which contains only one set of conjugate cycles of order k involves an invariant subgroup H such that H and each of the corresponding co-sets involves all these conjugate cycles, then H must be transitive and must contain only one set of conjugate cycles of order k.*

Since each of these cycles appears in H and in every co-set of G with respect to H, it results directly that it has as many distinct conjugates under H as it has under G. That is, all these cycles form a single set of conjugates under H. If H were intransitive, all of these cycles would therefore appear in one and in only one of its transitive constituents. This is impossible since the transitive constituents of H are transformed transitively under G.

It is now easy to prove that cycles of G which constitute a single set of conjugates must be of order 2. In fact, if these cycles were of prime order $p > 3$, the substitutions which would transform one such cycle into all its different powers less than p would involve cycles of composite order. It remains therefore only to consider the case when these cycles are assumed to be of order 3. In this case there would also be cycles of order 2 in G, and hence the order of G would be of the form $2^\alpha 3^\beta$. In particular, G would be solvable.

If G contains an invariant subgroup H of index 3 and all the cycles of order 3 are conjugate, each of the co-sets corresponding to the operators of order 3 in this quotient group involves all the different cycles of order 3 contained in G, and these cycles are also found in H. Hence it follows from Theorem II that H would be transitive and would involve only one set of conjugate cycles of order 3, while all of its other cycles would be of order 2. This subgroup being of the same degree as G could therefore be used in place of G.

By repeating this process we finally would arrive at a transitive group which would be of the same degree as G and could be used for G, and which would have no in-

variant subgroup of index 3. This group would therefore contain a subgroup of index 2 involving all of its cycles of order 3. If not all the remaining substitutions were of order 2, all the cycles of order 3 would again be conjugate under this invariant subgroup, and hence this subgroup would again be transitive and involve cycles of order 2. Since it may be assumed that all of the remaining substitutions would be of order 2, this subgroup would have the property that each of its substitutions could correspond to its inverse in an automorphism, and hence it would be abelian. Since it would also be transitive, it would be regular. This is clearly impossible since it has been assumed that $n > 4$. We have therefore proved the following theorem.

THEOREM IV. *The symmetric group of degree* 3 *is the only non-prime power transitive group which has the properties that it involves cycles of only two different orders and that all the different cycles of the larger order are conjugate.*

When all of the different cycles of order 2 contained in G form a single set of conjugates, it may be assumed that G involves no subgroups of index 2. In fact, if there were such a subgroup each of the remaining substitutions would have to be of even order, and hence each of them would involve one and only one cycle of order 2, while the subgroup of index 2 would involve no cycle of order 2, and hence it would be of odd order. Since this is impossible, and since G is solvable, it follows that G involves an invariant subgroup of index p, where p is an odd prime number. The order of G is therefore $2^{\alpha}p^{\beta}$.

THE UNIVERSITY OF ILLINOIS

ALGEBRAS AND THEIR ARITHMETICS*

BY L. E. DICKSON

1. *Introduction.* Beginning with Hamilton's discovery of quaternions eighty years ago, there has been a widespread interest in linear associative algebras, a subject known also under the name of hypercomplex numbers. The list of investigators in this field includes the following well known names: Hamilton, Cayley, Clifford, and Sylvester in England; Poincaré and Cartan in France; Weierstrass, Frobenius, Lipschitz, Molien, Scheffers, and Study in Germany; A. Hurwitz and Du Pasquier in Switzerland; Benjamin Peirce, C. S. Peirce, Taber, Wedderburn, Hazlett, and others in America.

Needed guides to the extensive literature on this subject are furnished by the recent book by Scorza and the two books by the writer. Many of the papers, especially the older ones, contain serious errors and obscurities. Again, a large proportion of the papers are now obsolete, since they either treat only special algebras or fail in an attempt to give a general theory, and especially since they deal only with algebras over the field of all complex numbers. But the results obtained for this very special case have since been extended to algebras over any field, and it is the latter general subject which is the really important one both for algebra and for the theory of numbers.

---

* A Report presented to the Society, by invitation of the program committee, at Cincinnati, December 28, 1923. This Report, the work of the author reported in it, and two other papers presented by him to the Society at the Cincinnati meetings (see this BULLETIN, page 280 of this issue) formed the basis for the award to the author of the seventy-fifth anniversary prize of the American Association for the Advancement of Science for the most notable contribution to the advancement of science presented at the meetings of that Association and its affiliated societies at Cincinnati on December 27, 1923 to January 2, 1924. (See this BULLETIN, vol. 30, Nos. 1–2, p. 90.) THE EDITORS.

We shall not attempt to give a complete survey of the entire subject of algebras, but shall restrict attention to a few results on the algebraic side which will be required in our account of the chief results in the recent remarkable theory of the arithmetics of algebras.

Instead of presenting the formal definition of a general algebra by postulates, we shall employ typical illustrations.

2. *Algebra of Complex Numbers.* A complex number $a \cdot 1 + bi$ is said to have the real *coordinates* a and b and the *basal units* 1 and i. All complex numbers form an algebra of *order* 2 over the field of all real numbers. We obtain another algebra by restricting the coordinates a and b to rational values; it is an algebra of order 2 over the field composed of all rational numbers. In general, a set of real or complex numbers is called a *field* if the sum, difference, product, and quotient (except by zero) of any two numbers of the set are also numbers of the set.

3. *Algebra of Matrices.* We shall now define a more typical algebra which plays an important rôle in our further discussion. Consider 2-rowed square matrices

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad \mu = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

whose elements $a, b, c, d,$ etc., are numbers of any chosen field F. We define the sum and the product of these matrices to be

$$m + \mu = \begin{pmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{pmatrix}, \qquad m\mu = \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ c\alpha + d\gamma & c\beta + d\delta \end{pmatrix}.$$

If k is any number of the field F, we call

$$km = \begin{pmatrix} ka & kb \\ kc & kd \end{pmatrix}$$

the *scalar product* of the number k and the matrix m. Consider the four special matrices

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then the foregoing matrix m can be expressed in the form

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} = ae_{11} + be_{12} + ce_{21} + de_{22}.$$

Hence the algebra formed of all 2-rowed matrices with elements in the field F has the four basal units $e_{11}, e_{12}, e_{21}, e_{22}$ and is of order 4.

Similarly, all n-rowed square matrices with elements in a field F form an algebra of order n^2, called a *simple matric algebra.*

In the definition of any algebra over any field F, we employ three operations called addition, multiplication, and scalar multiplication, which are assumed to have properties entirely analogous to those holding for the foregoing three operations on matrices.

4. *Quaternions.* The four special matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$k = ij = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

satisfy the relations

$$i^2 = j^2 = k^2 = -I, \quad ij = k = -ji, \quad ki = j = -ik,$$
$$jk = i = -kj,$$

and are the basal units of quaternions

$$xI + yi + zj + wk.$$

Since matrix I plays the rôle of unity, it is usually denoted by 1. Quaternions may therefore be obtained very simply from matrices. Only when the field F contains $\sqrt{-1}$ is the present algebra of quaternions over the field F the same as the foregoing simple matric algebra of order 4.

5. *Definitions.* Let F be a field all of whose numbers are real. Consider any quaternion $q = x + yi + zj + wk$ whose four coordinates belong to F and are not all zero. It is readily verified that the product of q by its conjugate

$q' = x - yi - zj - wk$ is $N = x^2 + y^2 + z^2 + w^2$. Since $x, \ldots, w$ are not all zero, we have $N \neq 0$. Evidently q has the inverse $q^{-1} = (1/N)q'$, which is a quaternion with coordinates in F. Then the equation $xq = r$ has the solution $x = rq^{-1}$, while $qy = r$ has the solution $y = q^{-1}r$. Hence our algebra of all quaternions over the real field F is an example of a *division* algebra, in which each of the two kinds of division (except by zero) can always be performed uniquely.

The special quaternions $x + yi$ form an algebra of order 2 called a *sub-algebra* of the algebra of all quaternions.

A sub-algebra I of an algebra A is called *invariant* in A if the product of every element of I by every element of A is an element of I, and if likewise the product of every element of A by every element of I is in I.

In case A has no invariant sub-algebra other than itself, A is called a *simple* algebra. It is a fundamental theorem that every simple algebra A is a direct product of a simple matric algebra and a division algebra D; this may be understood to mean that all elements of A can be expressed as matrices whose elements belong to D.

An element is called *nilpotent* if some power of it is zero. For example, the square of the foregoing matrix

$$e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

is the matrix zero all four of whose elements are zero, whence e_{12} is nilpotent.

An algebra is called nilpotent if all of its elements are nilpotent. A *semi-simple* algebra is one which has no nilpotent invariant sub-algebra.

An algebra A is said to be the *sum* of two sub-algebras B and C if every element of A can be expressed as a sum of an element of B and an element of C. If also the product of every element of B and every element of C is zero, and vice versa, and if B and C have in common

only the element zero, then A is called the *direct sum* of B and C.

Every semi-simple algebra is either simple or is a direct sum of simple algebras, and conversely.

The principal theorem on algebras states that every algebra which is neither nilpotent nor semi-simple is the sum of its unique maximal nilpotent invariant sub-algebra and a semi-simple sub-algebra.

6. *Evolution of the Arithmetic of Algebras.* We have now defined those terms and stated those theorems concerning algebras which are needed in our account of the arithmetic of algebras. That subject has been surprisingly slow in its evolution. Quite naturally the arithmetic of quaternions received attention next to the arithmetic of complex integers.

Lipschitz, in his book of 1886, called a quaternion integral if and only if its four coordinates are all ordinary integers. But his theory was extremely complicated; it was not a success since such integral quaternions do not obey the essential laws of divisibility of ordinary integers. For example, there does not exist a greatest common left divisor of 2 and $q = 1+i+j+k$. For, the only factors of 2 are the products of 2, 1, $1+i$, $1+j$ or $1+k$ by the various *units* ± 1, $\pm i$, $\pm j$, $\pm k$ (i. e., divisors of unity); while the only factors of q are the products of q, 1, $1+i$, $1+j$ or $1+k$ by the units. But no one of the four common factors listed is divisible by all of the others since $1+i$, $1+j$ and $1+k$ are indecomposable.

But A. Hurwitz in his memoir of 1896 and book of 1919 obtained a wholly satisfactory arithmetic of quaternions. He called a quaternion *integral* if its coordinates are either all integers or all halves of odd integers, and proved that the essential laws of divisibility of ordinary integers hold also for integral quaternions. In particular, there exists a greatest common left (or right) divisor; that of 2 and q is 2 since q is the product of 2 by the integral quaternion $\frac{1}{2}q$.

7. *Postulational Formulation.* Although Hurwitz stated his definition only for the case of quaternions, it may be formulated for any associative algebra A over the field of rational numbers. The integral elements of A are defined to be the elements which belong to a set of elements having the following four properties:

C (closure): *The sum, difference, and product of any two elements of the set are also elements of the set.*

B (basis): *The set has a finite basis* (i. e., *contains elements* $b_1, \ldots, b_k$ *such that every element of the set is a linear combination of the b's with ordinary integral coefficients*).

U': *The set contains the basal units of A.*

M (maximal): *The set is a maximal set* (i. e., *is not contained in a larger set having properties C, B, U'*).

Note that Lipschitz's integral quaternions with integral coordinates have the properties C, B, U', while Hurwitz's integral quaternions have also property M. That a maximal set is superior to other sets is in accord with the history of the evolution of our number system and our experience in various branches of mathematics.

Du Pasquier, a pupil of Hurwitz, published during the past 15 years many papers in which he modified Hurwitz's definition by replacing U' by the milder assumption U that the set contains the modulus 1 which plays the rôle of unity in multiplication.

8. *Former Definitions Unsatisfactory.* But the definitions by Hurwitz and Du Pasquier are both unsatisfactory in general. This fact will be illustrated for the special algebra having two basal units 1 and e, where $e^2 = 0$.

Under Du Pasquier's definition, any set of elements with properties B and U has a basis of the form $1, q = r + se$, where r and s are rational numbers and $s \neq 0$. Since q^2 belongs to the set by property C, we must have $q^2 = a + bq$, where a and b are ordinary integers. Thus

$$r^2 + 2rse = a + b(r + se), \quad r^2 = a + br, \quad 2rs = bs,$$

whence $2r = b$, $r^2 = a + 2r \cdot r$, $r^2 = -a$. Thus r is an

integer. We may therefore replace the initial basis $1, q$ by $1, q-r=se$. Our set, designated by $(1, se)$, is evidently contained in the larger set $(1, \frac{1}{2}se)$, which in turn is contained in the still larger set $(1, \frac{1}{4}se)$, etc., where each such set has properties *C, B, U*. In other words, there does not exist a maximal set, so that the algebra does not possess integral elements, and the definition of integral elements is unsatisfactory. In such a case, Du Pasquier suggested that we omit the desirable requirement *M* and define the integral elements to be those of an arbitrarily chosen one of the infinitude of sets $(1, se)$. But it has been definitely proved by the writer* that factorization into indecomposable integral elements is then not unique and cannot be made unique by the introduction of ideals however defined.

These insurmountable difficulties arise also under the definition by Hurwitz, which imposes on the foregoing sets $(1, se)$ the condition that s be the reciprocal of an integer, so that the basal unit e shall belong to the set.

9. *Final Theory. Rank-Equation Postulate.* The writer has recently published a satisfactory theory of the integral elements of any rational algebra in his book *Algebras and Their Arithmetics* (University of Chicago Press). He employs postulates *C, U, M* and (in place of *B*) the following assumption.

R (rank equation): *For every element of the set, the coefficients of the rank equation are all ordinary integers.*

If $\xi_1, \ldots, \xi_n$ are independent variables in the field of rational numbers, the element $x=\xi_1 u_1+\cdots+\xi_n u_n$ of a rational algebra A having the basal units $u_1. \ldots, u_n$ is a root of a uniquely determined *rank equation*

$$\omega^r+c_1\omega^{r-1}+\cdots+c_r=0$$

in which $c_1, \ldots, c_r$ are polynomials in $\xi_1, \ldots, \xi_n$ with rational coefficients, while x is not a root of an equation

* Bulletin, vol. 28 (1922), pp. 438-442; Journal de Mathématiques, (9), vol. 2 (1923), pp. 281-326.

of degree less than r all of whose coefficients are such polynomials. For example, the quaternion $\alpha+\beta i+\gamma j+\delta k$ and its conjugate $\alpha-\beta i-\gamma j-\delta k$ are roots of

$$\omega^2-2\alpha\omega+(\alpha^2+\beta^2+\gamma^2+\delta^2)=0,$$

which is the rank equation if α, β, γ, δ are independent variables in the field of rational numbers.

As a first justification of our new definition of the integral elements of any algebra A, note that for the case in which A is any algebraic field, there is a unique set of its elements which have properties C, U, R and M, and this set coincides with the totality of integral algebraic numbers of the field. In other words, the new theory is a direct generalization of the classic theory of algebraic numbers.

Next, the serious difficulties observed in the foregoing algebra with the basal units 1 and e entirely disappear under the new definition. For $x=a+be$, we evidently have $(x-a)^2=0$, which is the rank equation if a and b are independent variables in the field of rational numbers. Its coefficients are integers if and only if a is an integer. Evidently the unique maximal set of elements having properties C, U, R is composed of all the $x=a+be$ in which a is an integer and b is rational. These elements x are therefore the integral elements of the algebra. For any rational number k, the product of the integral elements $u=1+ke$ and $1-ke$ is 1, whence each is called a *unit*. Let $a\neq 0$ and choose $k=-b/a$. Then $xu=a$. The product of x by any unit u is said to be *associated* with x. Hence the integral elements are here associated with the ordinary integers a, so that the arithmetic of our algebra is associated with (and reduces to) the arithmetic of ordinary integers.

Note that our set of integral elements is the aggregate of the infinitude of non-maximal sets (1,*se*) of Du Pasquier. Our satisfactory set may therefore by derived by a suitable enlargement of any one of Du Pasquier's unsatisfactory sets. Similarly, Hurwitz obtained his satisfactory set of

integral quaternions by a suitable enlargement of Lipschitz's unsatisfactory set. There are many instances in the history of mathematics where success has been achieved by the principle of enlargement; examples are the growth of our number system and the introduction of ideals in the theory of algebraic numbers.

Note also that our integral elements $a+be$ do not have a finite basis, since b ranges over all rational numbers, and hence do not form a set of integral elements according to the definition either of Hurwitz or Du Pasquier. Thus the writer's conception of integral elements is entirely different from the conceptions of Hurwitz and Du Pasquier. It was only after long experimentation and tests of various kinds that the writer became fully convinced that his conception of the proper subject matter of arithmetics of algebras is from every standpoint wholly satisfactory and in particular far more desirable than all earlier conceptions. Moreover, the new conception greatly facilitated the development of a rich array of fundamental theorems, wholly lacking under former conceptions. The resulting remarkable science of the arithmetics of algebras furnishes the final justification of the new conception of the proper subject matter. It is obviously more difficult to justify a new determination of the proper subject matter of an embryo science than to compare different foundations of an established science.

10. *Theorems on Arithmetics.* According to the principal theorem on algebras stated above, any rational algebra A is a sum of its maximal nilpotent invariant sub-algebra N and a semi-simple sub-algebra S. The fundamental theorem on arithmetics states that the arithmetic of A is associated with that of S in the sense that every integral element (whose determinant is not zero) of A is the product of an integral element of S by a unit. For, all integral elements x of A are obtained by assigning values to the coordinates of those basal units which belong to S, such that the

S-component of x is an integral element of S, and assigning arbitrary rational values to the coordinates of the basal units which belong to N. Furthermore we can choose an element ν of N such that the product of x by the unit $1+\nu$ reduces to the component of x which belongs to S. In other words, the effect of multiplying x by a suitably chosen unit is to suppress the nilpotent component belonging to N (viz., *be* in our above example).

Next, the problem of the arithmetic of a semi-simple algebra S reduces to that of the arithmetics of simple algebras. For, we saw that S is a direct sum of simple algebras $S_1, S_2, \ldots$, so that each element σ of S is a sum of components $\sigma_1, \sigma_2, \ldots$ belonging to $S_1, S_2, \ldots$ respectively. It is an important theorem that when σ is an integral element of S, each σ_i is an integral element of S_i, and conversely. Moreover, the divisibility properties for S follow at once from those of the component algebras S_i.

We saw that the elements of any simple algebra Σ can be expressed as matrices whose elements range over the same division algebra D. It can be proved that the integral elements of Σ are those matrices whose elements range over the integral numbers of D, and conversely.

Let D be such that its integral numbers possess a division process yielding always a remainder whose norm is numerically less than the norm of the divisor. We may then establish a theory of reduction and equivalence of matrices whose elements are integral numbers of D. The resulting theory is a direct generalization of the classic theory of matrices whose elements are ordinary integers. In that case factorization into prime matrices is unique apart from unit factors. In our more general case, each matrix is the product of units and a diagonal matrix having exclusively zeros outside the diagonal, so that the arithmetic is associated with the simpler arithmetic of diagonal matrices.

We have therefore reduced the study of arithmetics of all rational algebras to the study of arithmetics of simple algebras, i. e., of matric algebras over a division algebra D,

and we have completed the latter study when D is of a certain type.

11. *Conclusion.* Under the new definition, any set of integral elements of the same order as the order of the rational algebra A has a basis if and only if A is semi-simple. Hence the new definition is in complete accord with the older definitions by Hurwitz and Du Pasquier only in the important case of semi-simple algebras. For the remaining algebras, the older definitions led to insurmountable difficulties, whereas under the new definition the arithmetic of such an algebra is associated with that of its semi-simple sub-algebra S, since we may suppress the components belonging to its maximal nilpotent invariant sub-algebra N. It is fortunate that we can get rid of these bizarre nilpotent components since they would interfere seriously in applications. Their elimination also greatly simplifies the theory.

The theory of algebraic numbers finds applications only to problems involving forms which contain only two variables homogeneously and hence can be factored into linear forms. This serious limitation may often be removed by employing hypercomplex numbers. For example, $x^2+y^2+z^2+w^2$ has as factors the quaternion $x+yi+zj+wk$ and its conjugate. Since the new theory of arithmetics of algebras finds applications to problems involving forms in any number of variables it furnishes us with an effective new tool for problems in algebra and the theory of numbers.

THE UNIVERSITY OF CHICAGO

FOUR BOOKS ON SPACE

Der Raum: Ein Beitrag zur Wissenschaftslehre. By Dr. Rudolf Carnap. Berlin, Reuter und Reichard, 1922. 87 pp.

Mathematik und Physik: Eine erkenntnistheoretische Untersuchung. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. 31 pp.

Die realistische Weltansicht und die Lehre vom Raume: Zweite Auflage; Erster Teil. By E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1923. x + 83 pp.

Mathematische Analyse des Raumproblems. Vorlesungen gehalten in Barcelona und Madrid. By Dr. Hermann Weyl. Berlin, Julius Springer, 1923. vii + 117 pp.

Kant described our knowledge of space and time as synthetic and a priori. By *synthetic,* he distinguished it from the analytic, more or less tautological judgments of abstract logic, while by *a priori,* he signified that it is independent of the concrete content of our senses. His account of the extensional properties of the universe was an attempt to bridge the gap between the purely abstract, ratiocinative science of geometry, and the obviously empirical nature of the space-world to which it applies.

While Kant's problem still exists, the last century has seen a tremendous change both in our notions of geometry and in our notions of the spatial world. Geometry no longer means Euclid, for since the days of Bolyai and Lobachevski we have become aware that there are other possible systems which yield no whit to the traditional geometry in the matter of logical rigor. The axioms of geometry signify no longer self-evident indubitable truths, but arbitrarily set assumptions. In short, from the mathematical standpoint, geometry is but a branch of logic, and like the rest of logic, is concerned with the consistency, the deductive sequence of its theorems, not with their truth. On the other hand, the universe is no longer treated as fitting primarily into the euclidean scheme, but into the more complicated schemes of the special or the general Einstein space-time system. For all this, the problem still remains as to how we can associate with our empirically known world of sense a mathematical structure which in at least its analysis situs properties is essentially that of Euclid, and in particular, how we can perform this association in a preliminary fashion, not merely as the final result of a long chain of careful experiments, but automatically, almost at first glance. This problem, as to the nature of our knowledge of space and time, has indeed become far more acute

because of the demands made by the Einstein theory on our imagination. It has led to a large and important literature, of which the books and pamphlets here reviewed form a significant part.

That of Carnap is written most distinctly from the standpoint of the professional philosopher, but shows a far deeper acquaintance than common with the mathematical and mathematico-logical advances of recent years. It is clearly written, and contains a good description of the various types of spaces and geometries recognized by the mathematicians, together with a valuable bibliography. Carnap tries to retain the spirit of the Kantian treatment of space, while discarding those details which the progress of the last century has shown to be indefensible. He distinguishes three levels of spatial knowledge, and three types of space to which they respectively pertain. The formal space of the mathematician is on the same epistemological basis as pure logic. It is known a priori, and there is nothing in its theorems that has not been put in by free assumption in its postulates. On the other hand, the space of the physicist is susceptible to an experimental investigation differing in no fundamental way from that by which we study the phenomena of heat or sound or electricity. Our knowledge of this space is inductive and a posteriori. Mediating between these two spaces is the realm where Carnap hopes to preserve what is of permanent value in the Kantian philosophy. This is the so called space of intuition, which is known as a condition of physical experience, independently of the amount of physical experience which we possess. The developments of the last century have made it impossible for us to attribute to metrical or even to projective geometry the importance of intrinsic properties of intuitional space, but (so Carnap maintains) the analysis situs properties of the universe are known as Kant considered all space to be known: a priori, but synthetically, and with more than purely logical content.

It is hard to believe that the analysis situs properties of the universe, even *im Kleinen,* are forever to be immune to a criticism of the type which has led to the theory of relativity. Relativity itself, indeed, questions our conventional notions of the *im Grossen* connectivity of the world. There are signs that the time may not be far distant when the atomicity which the quantum theory recognizes as a basal characteristic of the universe shall be referred to a fundamentally atomic conception of its space-time framework. Veblen indeed has put forward the suggestion that physics may come to describe the world by a modular space. The analysis situs of such a space—if it may properly be said to have any analysis situs—is immeasurably different from that with which we are familiar.

Like Carnap, Study attacks the problem of mediating between the deductive science of mathematics, which includes such parts of mathe-

matical physics as concern themselves with the formal development of the conclusions of assumptions not called into question in the course of the investigation, and the inductive science of experimental physics. Except that he is concerned with mathematics and experimental physics more broadly than as to their spatial aspects, Study's notion of the objects and methods of these disciplines does not depart widely from that of Carnap. The characteristic tool of the intermediate discipline, however, he conceives to be idealization. This idealization is of the nature of a *fiction*, a schematization, a diagram, which replaces the unmanageably complex properties of the empirical world by a working model of similar but simpler nature, satisfying the postulates of some known mathematical system. It is not of purely logical character, but involves a judgement of value, for similarity *in et per se* is an empty notion. It consists, as Study says, in dissecting out a single component of a complicated situation, and analyzing it apart from disturbing factors.

The theory of idealization certainly comes very close to the facts of our knowledge of space and of the external world. A not dissimilar view is held by Whitehead and by Russell. These authors have tried to develop the machinery by which we schematize the space and time of our experience—the space of bodies and the time of events—into the point-space and instant-time of theoretical physics. Study is unable to accept their treatment of this matter because of a fundamental epistemological difference.

Study is a realist. That is, he maintains that things have a reality entirely apart from their being perceived. Now, Russell is a realist also, but the realities of Russell are of the nature of sense-data, at least in large measure, and while they exist independently of being known, and may conceivably be shared by several observers, they are primarily given as objects of the experience of a single observer. In order to build out of these data the external world of physics, Russell makes the working hypothesis of the existence of other observers. Study points out that this hypothesis is of a precisely similar character to that of the existence of the external world of physics *tout simple*. Study accordingly starts his realism with this hypothesis. Like all hypotheses, he holds that it is capable of being confirmed or refuted by its consequences, whether it can be tested directly or not. It is clearly to be distinguished from a fiction, which is an assumption made for the purpose of conceptually simplifying a complicated situation, and with a full consciousness of its falsity.

Study, however, is fully appreciative of the value of Russell's analytical work. With all its incompleteness, its logical props and supports, it does represent a definite contribution to our understanding of what really is contained in our hypotheses concerning the universe. One can only regret that Study has made no similar attempt to dissect out the

precise content of the hypothesis of the existence of the external world. One thing at least is certain: that the hypothesis of the external world, taken as a simple and indivisible supposition, has no explanatory force whatever. It is only the hypothesis of a very special sort of external world which fills the need which Study finds. The nature of this external world needs elucidation and analysis of the Russellian type.

Study's *Die realistische Weltansicht und die Lehre vom Raume,* in addition to a clear and forceful exposition of his realism, contains much of a polemic nature. He combats the Neokantians, the Pragmatists, the Conventionalists such as Poincaré, and many schools besides. All of his discussion is valuable, for like Poincaré and unlike the generality of philosophers, Study has the firm basis of a real acquaintance *in concreto* with the universe which he analyses *in abstracto,* combined with a generous measure of dialectic skill. There are, however, places where one feels that the vehemence of his wrath might have been tempered with a little more regard for the amenities of scientific discussion. Nothing is gained by speaking in a contemptuous tone of the "*merkwürdige Psyche*" of William James.

Weyl's book brings us back again to the geometry of the mathematician. While he recognizes in his introduction the trichotomy of geometrical space, the world of bodies, and some third intermediate space of physics, he makes no attempt to elucidate the nature of this transitional realm. His purpose is purely to develop the structural characteristics of the space of mathematics, more especially in its differential aspects. While he avoids an explicit employment of the postulational method, his work is postulational in spirit. It consists in showing the conditions imposed on the metrical nature of space by certain suppositions of a very abstract and general character. His fundamental notion is that of *affiner Zusammenhang.* He conceives of space as consisting at every point of a sheaf of infinitesimal vectors. Given a system of coordinates, it is possible to relate the sheafs at neighboring points by means of the "parallel translation" of Levi-Civita. A method which enables us to select from all the possible parallel translations of P_1 to the neighboring point P_2 a particular "genuine" one, whatever P_1 and P_2 may be, determines what he calls the *affiner Zusammenhang* of space. In the case of the geometry of Riemann, where each linear element has once for all a length which is a quadratic form in its components, the *affiner Zusammenhang* of space is determined by the condition that genuine parallel translations preserve length *im Kleinen.* In Weyl's generalized differential geometry, where the lengths of two vectors at different points can only be compared by translating one to the other along some path, and in which the result of comparison depends in general on the path, the *affiner Zusammenhang* is still completely determined by the system of measurement.

Weyl takes up at some length his geometry, in which the lengths of the different vectors at each point are still given relatively by the square root of a quadratic form in their components, but the relative change in the length of a vector under a small displacement of its origin, its components remaining the same, is given as a linear form in the components of displacement of the origin.

Weyl exhibits differential expressions, respectively known as the *Streckenwirbel* and the *Vektorwirbel*, the vanishing of which gives the condition that space be Riemannian or euclidean, as the case may be. He also shows that the familiar non-euclidean geometries of Lobachevski and of Riemann are the only homogeneous metrical geometries: the only geometries, that is, in which there is no intrinsic metrical characteristic by which any two points may be distinguished. To this homogeneity he assigns a certain epistemological significance; namely, that if any particular metrical structure of space be given *a priori*, it must be a homogeneous one.

Up to this point Weyl has been recognizing the quadratic form as fundamental for the determination of a system of measurement. The question of course arises, "Why not a quartic or sextic form?" In the case of a homogeneous space, following Helmholtz, Weyl shows that the condition that free mobility about a point be possible limits our differential form of distance to the square root of a quadratic form. In the more general case of an arbitrary metrical space, he here proves for the first time the exceedingly important theorem that the only possible measure of the length of all infinitesimal vectors which determines invariantly the volume of an infinitesimal parallelopiped with a point P as vertex and which permits the unique determination of an *affiner Zusammenhang* by the condition that distances are preserved, is the square root of a non-degenerate quadratic form in the components of the vectors.

It will be seen that, like the postulationalists, Weyl is interested in penetrating to certain very general propositions which lie at the basis of geometry. He is fortunate, however, in not being bound in the pedantic straight-jacket of independence-proofs and postulate-counting which has strangled a most promising young science almost in its cradle. Weyl is not behind the postulationists in rigor, but he is far ahead of them in imagination, and he relegates his meticulous dissection of logical minutiae to the place where it belongs,—the back of the book.

NORBERT WIENER

TWO RECENT BOOKS ON ALGEBRAS

Algebras and their Arithmetics. By Leonard Eugene Dickson. University of Chicago Press, 1923. ix + 241 pp.

Corpi Numerici e Algebre. By Gaetano Scorza. Messina, Giuseppe Principato, 1921. ix + 462 pp.

On first picking up Professor Dickson's book, the reader's attention is caught by the fact that the author assumes no knowledge beyond the most elementary parts of a first course in the theory of equations. As the reviewer turned from one page to another, there was a certain ease and simplicity in the style which called to mind some of those science primers of the days of our grandfathers. One felt once more the half-forgotten glamor of the rainy day long ago when one discovered several of these "primers", during one's secret investigation of a pompous old book-case on the landing of the stair.

In the first chapter, he begins with the definition of a field of numbers and the elementary notions of a linear transformation and then easily makes the transition to matrices. Having pointed out that the set of all complex numbers, $a + bi$, is an algebra over the field of all real numbers and that the set of all p-rowed square matrices with elements in any field, F, is an algebra over F, in which multiplication is usually not commutative and in which division may fail, the author defines the general algebra over any field by five postulates which differ from those used in Scorza's book in one respect. After definitions of a few elementary notions in connection with algebras, he returns to matrices and shows that the matric algebra consisting of all two-rowed square matrices whose elements are complex numbers is carried by a linear transformation of the units into the familiar quaternions. Thus, at the very beginning, he gives an illustration of one of the most interesting and important results in the general theory of algebras, to wit, that any associative division algebra (an algebra in which division is always uniquely possible when the divisor is not zero) is equivalent, in a suitably enlarged field, to a matric algebra.

The second chapter is a brief one giving several elementary but useful theorems about linear sets, which are the same as algebras except that they are not necessarily closed under multiplication. Although the third chapter is also rather brief, it presents, in that simple, direct manner so characteristic of most of this author's work, about a dozen theorems and corollaries grouped around the fundamental notions of subalgebra, invariant subalgebra (analogous to self-conjugate subgroup in the theory of groups), reducible algebras and simple algebras. Of

these the most significant is the one which asserts that any reducible algebra with a modulus (i. e., a number which behaves under multiplication like unity) can be expressed as a direct sum of irreducible algebras each with a modulus in one and essentially only one way. Together, these two chapters contain many of the theorems used in the later proof of the fundamental theorem of linear algebras due to Wedderburn.

Chapter 4 is concerned primarily with nilpotent algebras and idempotent elements. A nilpotent element is one such that some power of it is zero, and an analogous definition applies to nilpotent algebras. Although such algebras may seem scarcely more than figments of the imagination, yet it is very easy to exhibit one such, to wit, the algebra consisting of a single unit, e_{12}, a square matrix of order two having unity in the first row and second column and zeros elsewhere, for $(e_{12})^2 = 0$. In fact, one wishes that the author had taken a few lines to give a simple example of this type to remove the scruples of the materialistic soul who insists on having everything in mathematics securely anchored to something tangible. After defining idempotent elements (i. e., numbers u such that $u^2 = u$) and properly nilpotent elements (Scorza's *elementi eccezionali)*, he proves Peirce's theorem about the decomposition of an algebra relative to an idempotent element. The last part of the chapter is concerned with semi-simple algebras—those containing no nilpotent invariant subalgebra. In addition to proving theorems which show that, under certain conditions, a subalgebra of a given algebra, A, is necessarily semi-simple, he shows that any semi-simple algebra which is not simple is the direct sum of simple algebras, and conversely. Although the results are necessarily technical, as are all results in mathematics, yet the author manages to keep the number of technical terms down to a minimum, so that, if the reader is willing to do any thinking, he will find that the text is most disarming in its simplicity.

One of the most fascinating chapters in the book is Chapter 5. This is devoted entirely to division algebras, which might be described as fields that are not necessarily commutative, if we use the word "field" (Körper) in Hurwitz' more general sense. In view of Wedderburn's theorem (proved in the next chapter) that every simple algebra over a field F can be expressed as the direct product of a division algebra over F and a simple matric algebra over F, a study of division algebras is exceedingly important. After eight elementary theorems and corollaries, each of which is proved in a very few lines, the author then devotes the remainder of the chapter to some important results on the determination of all division algebras. He first gives a simplification of his own elementary and neat proof of the famous theorem due to Frobenius and C. S. Peirce, to wit, that the only division algebras

over the field of reals are (1) the field of reals, (2) the field of complex numbers and (3) the algebra of real quaternions. The text then defines the type of division algebra D of order n^2 over a field F which were discovered by the author and called Dickson algebras by Wedderburn. For the purpose of this review it will suffice to say that they are natural generalizations of real quaternions and are intimately connected with the theory of abelian equations. If the reader is curious, we can but refer him to the text, where the author gives in simplified form the most important results of his very interesting memoir, *Linear associative algebras and abelian equations*, in the TRANSACTIONS for 1914, together with other results. At the end of the chapter is a brief summary in which he tantalizingly refers the reader to Appendices I and II for further results in the general theory of division algebras. These results were proved by Wedderburn in two articles in the TRANSACTIONS for 1914 and 1921 and demonstrate still further the importance of Dickson algebras.

While Chapter 6 may not appeal to the casual reader quite so much as the one on division algebras, yet to any one who has read the preceding chapters thoughtfully, the present chapter on the structure of the general algebra will appear as a thing of beauty. Here will be found Wedderburn's important theorem that essentially every simple algebra can be expressed as the direct product of a division algebra and a simple matric algebra; and conversely. By this theorem are determined all simple algebras over the field of reals. Finally, he proves a vital theorem which, when shorn of technicalities, essentially reduces the problem of the determination of all algebras to the determination of certain special types.

In the next chapter, the author views algebras from an entirely different standpoint. Hitherto, he has proved every theorem by the aid of Wedderburn's calculus of linear sets; but now he recurs to the older method by which an algebra is thought of as given by a multiplication-table. Here he proves Peirce's theorem that every associative algebra is equivalent to a subalgebra of a matric algebra. This is followed by the orthodox set of theorems on characteristic matrices, determinants and equations, and the rank equation.

Although the final chapter of the first part of the book is brief, it is of vital importance. The author picks up the thread which he momentarily dropped at the end of Chapter 6 and then, by using certain principles of Chapter 7, he proceeds to prove the principal theorem which asserts that any associative algebra over a non-modular field, F, can be expressed as the sum of its maximal nilpotent invariant subalgebra and a semi-simple algebra. Thus, in one hundred and twenty-seven pages, he gives the essence (the triple extract, as it were) of all that is interesting and important in the general theory of algebras.

While the first part of the book is largely Wedderburn's work recast by Dickson, the second part (on the theory of arithmetics) is entirely due to Dickson. Just as the theory of complex integers, due to Gauss, is in some respects equivalent to the representation of all substitutions, with rational coefficients, which leave unaltered a sum of two squares, so similarly the transformations which leave unaltered a sum of three squares lead to quaternions. Lipschitz, by the consideration of such substitutions having rational coefficients, was lead to consider integral quaternions and then irreducible or prime quaternions.

In his book,* he defines integral quaternions as those having all coordinates rational integers and then develops a theory of such quaternions by using a rather complicated theory of congruences. But, although his definition of an integral quaternion was in some ways a natural one, it led to a theory with strange contradictions. From this definition he proves that an integral quaternion has, in general, a set of irreducible factors which is, in a certain sense, essentially unique; but if the norm of the quaternion is divisible by 4, then the quaternion has not merely one but twenty-four sets of irreducible factors which are essentially distinct in any understanding of the term.

To Hurwitz belongs the credit of the discovery that the reason for this and other anomalies lay in the definition of integral quaternion used by Lipschitz. In his memoir in the GÖTTINGER NACHRICHTEN for 1896, Hurwitz gave a new definition which enlarged the set of integral quaternions and thus removed the above contradictions. But Du Pasquier, in a series of papers beginning in 1909, noticed that Hurwitz's definition is not satisfactory for every linear algebra and suggested other definitions. But these, in turn, are not suitable for every linear algebra, as they may either be vacuous or lead to insurmountable difficulties.

During the last few years, Professor Dickson has been interested in this problem, and his more important results are given in the second part of this book. In Chapter 9 he gives, in a simple and brief manner, the barest essentials of the theory of integral algebraic numbers; similarly, in Chapter 11, he gives the elements of the theory of fields, so that the reader who is unacquainted with these branches of mathematics can read the long and important Chapter 10, which gives Dickson's new work, with both understanding and appreciation.

In this chapter he defines an integer for any (associative) algebra having a modulus, over the field of rational numbers, in such a way that, when applied to the special case of quaternions, the new definition yields Hurwitz' set of integral quaternions and yet, when applied to

* *Untersuchungen über die Summen von Quadraten,* Bonn, 1886; French translation in JOURNAL DE MATHÉMATIQUES, (4), vol. 2 (1886), pp. 393–439.

the general algebra, it leads to no contradiction or difficulty. He thus proves that the arithmetic of an algebra A is known as soon as we know that of a certain semi-simple subalgebra, S. Similarly, the arithmetic of a semi-simple algebra, S, is known when we know the arithmetic of each of the simple algebras, S_i, of which it is the direct sum. In turn, the integers of a simple algebra are known when we know the integers of a certain division algebra, D. Thus, under the new definition, the problem of arithmetics of all algebras reduces to the case of simple algebras and finally, in large measure, to the case of division algebras. Moreover, the arithmetic of certain simple algebras is treated by generalizing the classic theory of matrices whose elements are integers. As a direct application, he obtains all integral solutions of some Diophantine equations which had not been solved completely previous to Dickson's solution.*

Taking it all in all, this most recent book from the hand of this indefatigable writer will be a source of keen joy to any one who feels that a composition in mathematics, like one in poetry, should have a fine balance between nicety of detail and a broad sweep that gives inspiration. It approaches very close to the French dictum, "Nothing can be added and nothing removed to improve the effect".

Professor Scorza's book is of a very different type. It seems a fair philosophy, both for life in general and for book-reviewing in particular, to judge the attainment of an individual by his purpose. Since the author says, in his preface, that this book is intended for young students in the university, one is inclined to expect a little more organization and conscious orientation than in a volume written for specialists. But the phrase that seems best to describe this book is "an encyclopedic treatise on the theory of algebras", using the term "algebra" in the broad sense as including theory of numbers, matrices, and hypercomplex numbers. The book was to have consisted of three parts: (1) theory of number fields, (2) general theory of hypercomplex numbers or linear algebras, (3) application of (2). But, he confesses, the second part grew so large that he was forced to limit the third part to an appendix of fifty pages. It is to be regretted that he did not include more of the applications, especially those that would show to the mathematician who is inclined to be skeptical of the virtues of any

* The only typographical errors noticed were the following:—

p. 53, label at top—Replace "§ 37" by "§ 39".

p. 60, l. 10—Replace "primitive element" by "primitive idempotent element".

p. 87, l. 5—Replace "$A-N$ is simple, also A is simple" by "A is simple, also $A-N$ is simple".

p. 178, equation (18)—Replace "$e_i = e_i$" by "$e_i^2 = e_i$".

abstract theory, the relation of hypercomplex numbers to quadric surfaces, abelian integrals, etc. Moreover, as will be seen below, the book would have gained both in power and charm if the second part had been compressed.

Part I, on the general theory of number fields (130 pages) is divided into three chapters. Of these, the first gives the definition of a number field via four postulates, the rudiments of the theory and the basic differences between finite and infinite fields. In this chapter he also introduces the notation for n-tuples and gives the elements of the theory of matrices and determinants. The second chapter is the longest of the three and is devoted to polynomials. After the usual definitions for polynomials in several variables, he gives the orthodox work on divisibility, rational fractions, the zeros of a polynomial and the solution of a system of linear equations together with Leibniz's formula for a general field (modular or otherwise), Galois' generalization of Fermat's theorem and Euclid's algorithm for the highest common factor. In the final chapter, he gives the fundamental properties of Galois fields.

The reader is soon struck by the fact that the author seems to want every definition and theorem expressed with a meticulous care that is rare. All through the text he makes precise statements similar to this: "Polynomials in the same field and in the same indeterminates that differ at most by null monomials are regarded as not distinct". At first, such precise care will please the cautious mathematician; but, after page upon page of such detailed precision, the reader begins to weary. In the case of some theorems, this results in his spreading over a page or more a proof that could be given in a few lines with complete rigor and greater simplicity. In fact, one is inclined to wonder if this style would not tend to dull the interest of the young student for whom the text is written.

Part II, on the general theory of linear algebras (273 pages), is divided into six chapters, of which the first gives in forty-seven pages not only the orthodox elementary work with matrices, but also much interesting material not usually found in a text-book (such as Kowalewski's *Determinantentheorie*) of which the most important are Cayley's theorem that every square matrix satisfies its characteristic equation, Frobenius' beautiful theorem that the equation of lowest degree satisfied by a square matrix is a factor of the characteristic determinant obtained in a certain definite manner and Hadamard's well known result for the maximum value of a determinant.

The remaining chapters of this part are devoted to the general theory of algebras and look much like a revision of the first part of the author's memoir *Le algebre di ordine qualunque e le matrici di Riemann* which appeared in the RENDICONTI DI PALERMO for 1921. In many sections, the point of view and the method are those of Wedderburn's beautiful

memoir in the PROCEEDINGS OF THE LONDON SOCIETY for 1908, though Professor Scorza has polished a number of the proofs—some to a slight degree and some to a greater degree, notably the proof of the theorem on the structure of simple algebras.

Although Chapter 2 is, with one exception, the longest in the book, it contains scarcely more than a long array of definitions and minor results in the general theory of linear algebras. To one already familiar with the essentials of the theory, the significance of this chapter will, of course, be clear; but the reviewer wonders if, when these multitudinous minor results are presented at the beginning of the theory, in a terminology unnecessarily complicated, they may not confuse or discourage some students. Chapters 3 and 4 are brief and discuss properties of an algebra connected with its invariant subalgebras and idempotent elements, respectively. Of these, the most important is the theorem that any algebra is the sum of its maximum nilpotent subalgebra and a semi-simple algebra (page 255). Chapter 5 is also comparatively brief and is concerned with the coordinates of an algebra, characteristic determinants, etc. The last chapter is the longest, occupying nearly one fifth of the entire book. In this he proves the set of theorems on the structure of the different types of algebras: simple algebras, matric algebras, division algebras; but ignores the very interesting and important work on division algebras by Dickson and Wedderburn referred to above in connection with Dickson's book.

Although the subject-matter of these five chapters is essentially the same as that of the first half of Dickson's book, it would be scarcely possible to imagine two mathematical books more different in their treatment. This difference is partly due to the unnecessary detail in Scorza's presentation, and partly due to the unfortunate fact that Scorza uses the older terminology which has been discarded. Also, it sometimes seems to the reader that he delights in multiplying the number of technical terms. For example, since his *algebra eccezionale* of an algebra A is simply the maximum nilpotent invariant subalgebra of A, why not call it such? The latter term is not only more descriptive, but is also the usual one.

At the end he has placed a bibliography of fifty-three books and memoirs on number fields, matrices and linear algebras. Although this is surely far better than none, it is not all that might be desired, since there are several errors, both of omission and commission. For example, why does it omit

Steinitz, Ernst, *Algebraische Theorie der Körper*, JOURNAL FÜR MATHEMATIK, vol. 137 (1909), pp. 167–309;

Dickson, L. E., *Linear associative algebras and abelian equations*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 31–46;

Wedderburn, J. H. M., *A type of primitive algebra*, ibid., pp. 162–166?

The last two articles (together with a brief but suggestive paper by Wedderburn in the TRANSACTIONS for 1921 which, presumably, appeared too late to be included in the bibliography) are, without doubt, the most interesting and the only important articles on the general theory of division algebras, and contain all that has been discovered about this difficult and extremely fascinating branch of linear algebras since Wedderburn's memoir in the PROCEEDINGS OF THE LONDON SOCIETY mentioned elsewhere in this review.

One notes with dismay that there is no index. This lack is an inconvenience in any book of this size and is only slightly ameliorated by putting in bold-face type the caption of every section and of every definition. In fact, one might almost say that the presence or lack of an index is a characteristic invariant which distinguishes Anglo-American texts from Continental ones.*

But altogether, the book contains a great deal of information not previously available outside of technical periodicals; and Professor Scorza is to be congratulated on the courage with which he attempted and the care with which he finished the task that he had set himself. As one reads, one can not but feel that the task has been to him a pleasant one and that when it was completed, he left it with a caress.

OLIVE C. HAZLETT

---

* In addition to the errors listed in the *errata* at the end of the volume, the reviewer noted only the following non-trivial mistakes:

p. 118, l. 20—Interchange "Γ_1" and "Γ".

p. 167, l. 20—Replace "$\| \xi_{j,1} - \delta_{j,1}\, \sigma \| = 0$" by "$| \xi_{j,1} - \delta_{j,1}\, \sigma | = 0$".

p. 374, l. 4,6—Place square brackets around each "u".

A CORRECTION

In line 16, page 4, volume 29 of this BULLETIN (Jan., 1923), insert the words *and contain only decomposable continua* between the word *points* and the word *then*. The desirability of this correction was called to my attention by Professor J. R. Kline.

G. A. PFEIFFER

SHORTER NOTICES

Prolegomena to Analytical Geometry in Anisotropic Euclidean Space of Three Dimensions. By Eric Harold Neville. Cambridge, University Press, 1922. 22 + 368 pp.

The author's prolegomena are inspired by his feeling that to "let" the coordinates be complex instead of real is an indefensible absurdity. In order to justify the use of complex space he states and develops definitions, principles, and especially a vocabulary appropriate to real analytic geometry, builds a complex space, proves it to be unique, and then invests this ideal space with a geometry whose vocabulary he has previously developed. A word as to the title is necessary. In a euclidean four-space there are *isotropic* three-spaces, isotropic in the sense of the word as introduced by Laguerre. The study of the geometry of these isotropic three-spaces forms no part of the discussion.

The treatise is divided into five books. Books I–III deal with the avoidance of ambiguity in the measurement of angles and with projection, with vector analysis, and with Cartesian and vector frames. Book IV is devoted to the construction of algebraic space, the fundamental axioms involving vectors. The book concludes with a proof that complex space is unique. Especial attention is given here and later to the isotropic plane; a table of paragraphs relating to the isotropic plane is a welcome aid. Book V is devoted to a discussion of the geometry of this ideal space with special chapters on curves and surfaces especially conics and conicoids, on circles, and on spheres.

By the exercise of great care and ingenuity in his choice of definitions the author is enabled to state many important theorems without exceptions. Everywhere exceptional or degenerate cases are adequately covered. The treatment is rigorous, thorough, and careful. This care extends to the proof-reading. Only two slips of any kind were noted: (*a*) in 221.41, (*b*) in a contradiction between 553.49 and 562.53. In paragraph 445 the significance of geometric theorems relative to complex space is summarised in the clearest, most incisive manner that the reviewer has ever seen.

On the other hand, many readers will find that the subject-matter generally is, frankly, not inspiring. The chapters on vectors will attract some, the chapters on isotropic planes others, the frivolous (and others) will rejoice over the humor of the preface and foot-notes; but despite the elegance of this book, it is doubtful if many readers will be attracted to a serious perusal.

B. H. Brown

Die Mathematischen Hilfsmittel des Physikers. By Dr. Erwin Madelung. Berlin, Julius Springer, 1922. 247 pp.

This book gives a tabular list of the mathematical results needed in theoretical physics. It is very comprehensive, and covers everything that is not of a distinctly special character. The book also contains a second part devoted to formulating the fundamental laws and principles of physics, and discussing their formal aspects very briefly. Originally, this latter was intended merely for the purpose of illustrating the use of some of the mathematics given in the first part, but the author developed it so that it has a certain completeness. In our opinion, this part is coordinate in importance with the first, so that the book is decidedly more than its name implies.

We give a very reduced extract from the table of contents:

Algebra: Lineare Gleichungssysteme. Matrizes und Determinanten. Kombinatorik (8 pp.). *Funktionen*: Allgemeine Funktionentheorie. Spezielle Funktionen (35 pp.). *Reihen* (9 pp.). *Differential- und Integralrechnung* (8 pp.). *Differentialgleichungen*: Allgemeines über Differentialgleichungen. Gewöhnliche Differentialgleichungen. Partielle Differentialgleichungen (37 pp.). *Lineare Integralgleichungen* (3 pp.). *Variationsrechnung* (5 pp.). *Transformationen*: Allgemeines über Transformationen. Koordinaten-Systeme und Koordinaten-Transformation. Berührungstransformation (22 pp.). *Vektoranalysis*: Koordinatenfreie Formulierung der Vektoranalysis. Koordinatenmässige Formulierung der Vektoranalysis in n-dimensionalen Räumen (28 pp.). *Wahrscheinlichkeitsrechnung und einige Physikalische Anwendungen* (10 pp.) *Mechanik*: Prinzipien der Mechanik. Einzelne Massenpunkte. Systeme von Massenpunkten. Starrer Körper. Mechanik der Kontinua (22 pp.). *Elektrizitätslehre* (16 pp.). *Relativitätstheorie*: Lorentztransformation. Elektrodynamik. Dynamik der Masse. Allgemeine Relativitätstheorie (14 pp.). *Thermodynamik*: (including the three laws, "phases", "mass-action", etc. 24 pp.).

The treatment of elliptic functions and integrals is characteristic of the book: periodic properties, Liouville's theorem, etc., are stated without proof; while all the steps of the reduction of the general integral

$$\int F(x, \sqrt{\alpha + \beta x + \gamma x^2 + \delta x^3 + \epsilon x^4})\, dx, \quad F \text{ rational}$$

to the three standard forms are given. Apparently the great concentration of the material has made it impossible to state all the restrictions of continuity, etc., in every case (notably in the definition of *analytic function*, which is not satisfactory).

This book will exert a broadening influence on both mathematician and physicist. Although it is intended as a table rather than a textbook, it will lead to a desire for breadth of knowledge.

B. O. Koopman

Grundzüge der Mehrdimensionalen Differentialgeometrie in direkter Darstellung. By D. J. Struik. Berlin, Springer, 1922. 198 pages.

In addition to the research papers which are published in any subject, there comes a time when one feels the need of a book which unifies the whole. The differential geometry of hyperspace has reached this point. Practically the only books on the subject are Killing's *Nichteuklidische Raumformen* and chapters in Bianchi's *Geometria Differenziale* so that the appearance of a book devoted entirely to differential geometry of n-dimensions is most welcome. This subject can well be said to have started with Riemann, followed by Christoffel, Beltrami, Lipschitz, and others who devoted a great deal of time to the study of the quadratic differential forms in n variables. These works were published between 1860 and 1880. In the latter eighties, Ricci and Bianchi began their work on n-dimensional geometry, and they and their followers have continued the work to the present time.

Ricci began his work in the absolute calculus in 1887, and he and Levi-Civita developed the subject almost to its present completeness many years ago, but it attracted but little attention. In fact it might be said that their work was practically unknown prior to 1913. Bianchi does use the covariant derivative as a notation but gives no indication of the powerful tool which Ricci made of it. However, when Einstein wrote his theory of gravitation and used the absolute calculus in its development, the subject became a live one and since that time there has been a whole army of mathematicians, scattered all over the world, working on it. The name which Ricci used was changed by Einstein to tensor calculus, and many people today are almost in total ignorance of what Ricci and Levi-Civita have done. It was, therefore, pleasant to find that Struik has dedicated his book to Ricci.

One of the outstanding features of the book is the "direct development". By introducing a sort of vector notation his work is freed from the dependence on the particular coordinate system used. The multiple algebra of the subject is developed to some extent but when one finds so many products used, it is quite a task on the memory of the reader to keep them all in mind. The notation, however, does allow one to sidestep the great mass of summation signs used by Ricci. The Clebsch-Aronhold notation is used to write forms of higher degree than the first as symbolic products of forms of the first order. Ricci's term "system" is replaced by "Affinor", and tensor is defined as a symmetric affinor. The first chapter is devoted to the multiplication of "affinors" both covariant and contravariant and mixed. In the second chapter the covariant derivative is developed and applied. The notion of geodesic parallelism introduced by Levi-Civita is made the basis of the work of this chapter. A differential is defined which bears the

same relation to an ordinary differential that geodesic parallelism bears to ordinary parallelism and from this the operator ∇ is defined similar to the usual definition. The absolute derivative is then derived as ∇V where V is a vector or system of the first order. The various products of ∇ with the various affinors give the formulas needed in hyperspace. Chapter three deals with the curvature properties of a curved space of m dimensions contained in a curved space of n dimensions which do not depend on the Riemann-Christoffel tensor. The whole development is similar to the ordinary treatment of an m-spread in a euclidean space of n dimensions. There is a remarkable similarity in the formulas in the two cases.

The fourth chapter treats those curvature properties which depend on the Riemann-Christoffel tensor.

The text is preceded by an introduction of twelve pages which it is worth anyone's while to read. The last twenty pages are devoted to a bibliography which contains nearly four hundred titles. This gives one a good notion as to the historic development of the subject. The author has been very careful to give credit to the proper author for all ideas and formulas. The reference one meets most frequently is to Schouten, for the notation used is largely due to him.

The book is not easy reading, but as one becomes more familiar with the notation, he will find that the difficulty decreases, and he will feel amply repaid for his trouble.

C. L. E. Moore

A First Course in Nomography. By S. Brodetsky. London, G. Bell and Sons, Ltd., 1920. 135 pp.

Brodetsky has supplied us with the sort of book the subject has needed: a brief, readable exposition of elementary character. It will well serve two purposes. It is an excellent introduction to D'Ocagne, and it also enables its reader to gain a practical working knowledge of the nature and uses of nomograms with a minimum expenditure of time.

The author states in his preface that "it is the object of this First Course to offer a clear and elementary account of the construction and use of such (nomographic) charts" and that "it is a treatment that should be found useful by the reader who desires to become acquainted both with the theory of nomography and with its practical use."

On the whole the author has carried out his intentions fairly well, but, as a text for college students, the book is open to criticism. While a considerable amount of knowledge of algebra, trigonometry, and analytic geometry is presupposed, the author's treatment is very uneven in its demands upon the reader's knowledge and intelligence.

While simple matters of mechanical detail are carefully explained, the theoretical discussions contain many serious gaps and fail to make skilful use of the knowledge the reader is supposed to possess.

Moreover, the reader's confidence in the author is severely shaken by a piece of grossly illogical reasoning on pages 37 and 38. The author asks: "Is it possible to subtract in such a way that the result is given on a scale graduated with the same unit as the one begun with?" His answer is: "No! If we examine Fig. 4 and Fig. 14 (the only two nomograms which give the result of subtraction, the former by $b = x - a$, the latter by $x = a - b$) we see that in each case the final unit is different from the first unit." Not only is the logic faulty, but the conclusion itself is incorrect. Fig. 19 on page 35, although designed to be used for the addition $x = a + b$, obviously is equally useful for the subtraction $a = x - b$, and the unit for a is the same as the unit for x. Hence the book, although very useful for some purposes, cannot be regarded as wholly satisfactory.

R. D. BEETLE

Un Théorème de Géométrie et ses Applications. By Georges Cuny. Paris, Vuibert, 1923. 6+102 pp.

Let us denote the cross ratio of the four lines OX, OY, OM, ON by R_M. Let OX and OY meet an algebraic curve C_n in $A_1, \ldots, A_n$, and $B_1, \ldots, B_n$, respectively. Let C_n meet an algebraic curve C_p in $\alpha_1, \ldots, \alpha_{np}$. Let C_p meet $A_i B_i$ in $\beta_1, \ldots, \beta_{np}$. The theorem is that the product of the R_α's is equal to the product of the R_β's. As a special case, if C_n has a multiple point of order $(n-1)$ at 0, this product is a function of p alone.

No one can fail to be impressed by the bewildering array of well known theorems which appear as special cases of this remarkably general theorem. As examples we note two general theorems on algebraic curves due to Newton, Pascal's hexagon theorem, Pappus' theorem, Carnot's theorem, Poncelet's theorem on the intersections of a cubic and conic, Frégier's theorem, Cazamian's theorem; examples selected from among scores of familiar projective and metric theorems. The methods of polar reciprocation, of inversion, and Laguerre's projective definition of angle are most happily employed. Extensions to three dimensions are briefly indicated.

The book should be of great value, especially to the younger graduate student. The reviewer believes that the possibilities of the theorem are by no means exhausted. Further investigations might well be incorporated in masters' theses. The subject matter is admirably adapted for student lectures in a course on modern geometry.

B. H. BROWN

Vector Analysis and the Theory of Relativity. By Francis D. Murnaghan. Baltimore, The Johns Hopkins Press, 1922. x + 122 pp.

This book is a brief exposition of what is commonly called the mathematics of relativity. The term vector analysis is used in a broad sense as equivalent to tensor analysis. Addressing himself primarily to the physicist the author connects up the new concept with those parts of mathematics with which the physicist may be assumed to be familiar. Thus after defining non-metrical n-dimensional space and spreads therein, the subject of tensors is approached through the consideration of integrals over such spreads which are independent of the parameters in terms of which the spread is given. After tensor algebra has been developed, it is applied to the proof of a generalized form of Stokes's lemma.

Metrical space and its attendant tensors are next considered and the relation between tensor analysis and the ordinary vector analysis noted. The resolved part of a tensor in an arbitrary direction is found and application made to a proof of a generalized form of Green's Theorem and to a discussion of Maxwell's equations.

The fifth chapter is a digression which the author advises be omitted in a first reading. It deals with the connection of tensor algebra with integral invariants and application to the statement of Faraday's law of moving circuits. It serves as another example of the fact emphasized in the book, that the methods of tensor analysis are by no means limited in their physical application to the relativity theory.

In the next chapter come covariant differentiation, the Riemann four index symbol, and Einstein's gravitational tensor. And finally the last chapter leads from the general tensor equations for the gravitational tensor to their solution in terms of a particular coordinate system for the field of a single particle and thence to the motion of the perihelion of Mercury and the bending of a light ray which grazes the sun. This jump from the general to the specific, which in so many of the works on relativity seems fraught with danger to life and limb, is here presented with a clarity and definiteness that delights the heart. Frequent examples involving the transformation from rectangular cartesian to space polar coordinates are used to illustrate the tensor analysis.

To the reviewer the approach to the tensor through integrals over a spread which are invariant under all changes of the parameters giving the spread is not as simple and direct as that through the differential properties of the spread. But the book is not an introduction to the relativity theory, and if we assume the reader already introduced to the theory we probably may assume a bowing acquaintanceship with its mathematics, and if this is the case the new point of view here developed will give him added insight into tensor analysis.

CHARLES E. WILDER

Dimensional Analysis. By P. W. Bridgman, New Haven, Yale University Press, 1922. ii 112 pp.

The substance of this book was given as a series of five lectures delivered at Harvard University in the spring of 1920. It treats of the *principle of similarity* used so effectively by the late Lord Rayleigh. A physical equation, according to the usual view, must express an equality between magnitudes of the same character, i. e., of the same dimensions. Nevertheless the author writes on page 42 the equation $v+s=gt+\frac{1}{2}gt^2$ where s is a length and v a velocity. But a complex equation of this type requires a special definition of the $+$sign and we cannot agree with the statement that "it is in itself a refutation of the intuitional method of proof of the principle of similarity." There is a useful chapter on the theory of model experiments and a collection of thirty-two examples. We feel that the book would benefit by expansion. Thus brief statements such as "Freedom from viscosity and complete turbulence of motion are seen by the analysis to be the same thing" (p. 85) are difficult to understand.

F. D. MURNAGHAN

Denken und Darstellung; *Logik und Werte*; *Dingliches und Menschliches in Mathematik und Naturwissenschaften.* Von E. Study. Braunschweig, Friedrich Vieweg und Sohn, 1921. 43 pp.

This pamphlet is interesting and valuable as expressing the views of a leading mathematical investigator on the ideals of mathematical teaching, study, and research. It is a polemic against the too formal notions of M. Pasch, who apparently regards the logical minuteness which is the final achievement of the analyst as the chief and almost the sole criterion of the value of a piece of mathematical work and as the goal to be striven for in mathematical education. Study rightly emphasises that this painstaking logical detail is merely a means for the avoidance of error, not an engine for the discovery of truth, that imagination is the first virtue of the mathematician, and that logical consistency alone is but sterile. He strongly dissents from Pasch when the latter separates a finer mathematics in which the utmost of logical precision is requisite from a coarser mathematics in which a moderate amount of fallacy is permissible. He demands strict logic everywhere, but a logic combined with imagination, not bound in the straight jacket of a dead syllogistic form. He says much that is worth reading about the bases of an evaluation of pieces of research, together with the ideals which should govern mathematical publication. The book stands beside those of Poincaré as one of the best expositions of how mathematical work is actually done, by a man whose great services to mathematics enable him to speak with authority.

NORBERT WIENER

Relativity, A systematic Treatment of Einstein's Theory. By J. Rice. London, Longmans, Green and Co., 1923. xvi+397 pp. $6.00.

The Mathematical Theory of Relativity. By A. Kopff. Translated by H. Levy. New York, E. P. Dutton (printed, however, in Aberdeen). viii+212 pp. $3.20.

These two books illustrate very well indeed the two tendencies of the flood of books on relativity that is now flowing from the presses of British publishers. On the one hand, we have a few books of great value, such as Eddington's *Mathematical Theory of Relativity.* On the other hand, we have innumerable translations of German expositions of *"Einstein"* that pay no attention to current progress in this field.

Mr. Rice's book is of the first type. In spite of a few instances of poor proof-reading, it contains a somewhat novel treatment of the restricted theory, an excellent development of the general theory, and a most valuable comparative study of the work of Einstein, Weyl, and Eddington. In an addendum, it contains a brief but interesting account of Einstein's recent amendment to his general theory, which appeared in the Berlin SITZUNGSBERICHTE of March, 1923. In short, this volume would be a valuable addition to the library of anyone interested in the problems on which Einstein and others have made noteworthy progress.

The volume by Kopff is an English translation of one of a multitude of German works on *"Einstein"* rather than on *relativity.* Kopff has presented the now classical theory in a scholarly manner. The book needed, however, an editor as well as a translator, the English edition being full of footnotes referring exclusively to German sources. For those to whom this work would be of value, the German edition would be a less expensive volume of equal value.

C. N. REYNOLDS, JR.

Die Buntordnung. By Arnold Kowalewski. Heft 1. *Entstehung und mathematischer Ausbau der Buntordnungslehre.* Leipzig, Wilhelm Engelmann, 1922. 53 pp.

The subtitle of this work describes it as "mathematical, philosophical, and technical considerations concerning a new tactical idea". The new idea is as follows: To arrange the $\left|\begin{smallmatrix} n \\ p \end{smallmatrix}\right|$ combinations of n things (elements) taken p at a time in a row in such a way that every set of k successive p-ads in the row shall contain no common element. Such an arrangement the author calls a diversified row (Buntreihe) of degree $k-1$. If such a row of degree $k-1$ is possible, while one of degree k is not possible, the row is called the "most diversified" for the given n and p. For given n and p the problem is naturally to find the most diversified rows. The solution of this problem is said to be of importance in experimental psychology (where the problem was indeed suggested) and in many other domains, including analysis situs.

The idea seems to the reviewer not without interest. Unfortunately the present booklet leaves the reader in doubt as to how far a general theory has or can be developed. The author confines himself here to a popular exposition of the more elementary aspects of the problem for small values of n and p and gives numerous special examples of diversified rows, diversified rings, etc. For a more complete account of his investigations he refers to a series of seven articles published in the SITZUNGSBERICHTE DER WIENER AKADEMIE DER WISSENSCHAFTEN beginning in 1915. The author, who is apparently a psychologist, should not be confused with the mathematician Gerhard Kowalewski.

J. W. YOUNG

Lehrbuch der Analytischen Geometrie. Zweiter Band: *Geometrie im Bündel und im Raum.* By Lothar Heffter. Leipzig and Berlin, B. G. Teubner, 1923. xii + 423 pp.

The first volume of this text on analytic geometry appeared in 1905, with L. Heffter and C. Koehler as joint authors. (See this BULLETIN, vol. 13, pp. 247–249.) In the preface to the second volume, the author expresses regret that the original plan of the first volume was modified and suggests that the first twenty articles be supplemented by his pamphlet, *Die Grundlagen der Geometrie als Unterbau für die analytische Geometrie* (1921).* He also states that in the present volume he has been especially concerned in trying to clear up the ambiguity that is often associated with the word "metric".

This volume opens with the geometry of the totality of lines and planes through a point. This completes Part II, *Geometry of two dimensions,* which was begun in volume I. Geometry in space of three dimensions occupies the remaining five-sixths of the book. The first seven chapters (125 pages) deal with projective geometry. Projective point and plane coordinates in space, projective theorems concerning points and planes, and projective coordinates of the straight line in space precede the general and special projective properties of the surfaces of the second order and second class and polarity. "Affine" geometry is disposed of in four chapters (75 pages) before the treatment of "Aquiform" geometry (seven chapters, 144 pages). This includes the principal axes and principal planes of surfaces of the second order, focal properties of central surfaces of the second order and of the paraboloid, systems of these surfaces, and biquadratic space curves.

Groups of exercises are inserted at intervals throughout the book. There are some references and there is a good index. The many figures are well drawn, and the subject matter is clearly presented.

E. B. COWLEY

* See this BULLETIN, vol. 28, p. 224.

NOTES

The third number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *On the potential of a homogeneous spherical cap, of a magnetic shell, and of a steady current,* by C. De Jans; *On cyclic-harmonic curves,* by Harold Hilton; *Multiple integrals in n-space,* by Philip Franklin; *On symmetric forms in n variables,* by Arnold Dresden; *Algebraic fields,* by J. H. M. Wedderburn; *A theorem concerning certain unit matrices with integer elements,* by H. R. Brahana.

At the annual meeting of the American Association for the Advancement of Science, held at Cincinnati December 27, 1923, to January 2, 1924, Dr. J. McKeen Cattell, editor of SCIENCE, was elected president. Vice-presidents were elected as follows: mathematics, Professor J. C. Fields; physics, Dr. E. F. Nichols; astronomy, Professor J. A. Miller; historical and philological sciences, Professor L. C. Karpinski. Professor T. H. Hildebrandt was elected as member of the Committee for Section A (mathematics). Professor G. A. Miller delivered his address as retiring vice-president of Section A before the joint session of that section with the American Mathematical Society and the Mathematical Association of America; the address was on *American mathematics during three-quarters of a century.* At this session Professor L. E. Dickson delivered, at the request of the program committee of the American Mathematical Society, an address on *Algebras and their arithmetics,* for which (together with his two other papers read before sessions of this Society) he was awarded the Association's prize for the most valuable contribution to science presented during the meeting (see this BULLETIN, vol. 30, Nos. 1-2, p. 90. Professor Dickson's address appears in full in the present issue of this BULLETIN on pages 247-257. A review of his book of the same title appears on pages 263-270.

Professor H. L. Rietz has been elected president of the Mathematical Association of America, and Professors J. L. Coolidge and Dunham Jackson vice-presidents.

Professor W. H. Young has been elected president of the London Mathematical Society. Professors L. N. G. Filon, Harold Hilton, and A. E. Joliffe vice-presidents, Dr. A. E. Western treasurer, and Professors G. H. Hardy and G. N. Watson secretaries.

The fiftieth anniversary of the founding of the Société Française de Physique was celebrated at Paris in December, 1923. On this occasion a general meeting of the International Union of Physics was held, on December 10.

The International Mathematical Congress to be held in Toronto on August 11–16, 1924 has been organized into the sections listed below. An Editorial Committee has been appointed, of which Professor J. C. Fields in the general chairman, and of which the American members for the several sections appear under the section heading below.

SECTION I: *Algebra, Theory of Numbers, Analysis.*
Professors D. R. Curtiss and A. J. Kempner.

SECTION II: *Geometry.*
Professors A. B. Coble and L. P. Eisenhart.

SECTION III: (a) *Mechanics, Mathematical Physics.*
(b) *Astronomy, Geophysics.*
Professors G. D. Birkhoff and F. R. Moulton.

SECTION IV: (a) *Electrical, Mechanical, Civil, and Mining Engineering.*
(b) *Aeronautics, Naval Architecture, Ballistics, Radiotelegraphy.*
Professors E. R. Hedrick and A. E. Kennelly.

SECTION V: *Statistics, Actuarial Science, Economics.*
Professors T. M. Putnam and H. L. Rietz.

SECTION VI: *History, Philosophy, Didactics.*
Professors R. C. Archibald and L. C. Karpinski.

The general Organizing Committee (Professor J. C. Fields, *Chairman*; Professor J. C. McLennan, *Acting chairman*; Professor J. L. Synge, *Secretary*) will welcome abstracts of papers offerred for presentation before the Congress; these should be sent to Professor Synge in care of the Royal Canadian Institute, Toronto. Among the noted Europeans who have indicated an intention of attending the Congress are Borel, de la Vallée Poussin, Eddington, Enriques, Fréchet, Hadamard, Koenigs, Marconi, Parsons, Pincherle, Sierpinski, Villat, Volterra, Young, Yule.

The Paris Academy of Sciences announces the award of the following prizes for 1922: the Bordin prize to Professor Emile Gau, of the University of Grenoble, for a memoir on partial differential equations of the second order integrable by Darboux's method (see this BULLETIN, vol. 27, p. 239); the Francœur prize to the Abbé Gaston Bertrand, for his work in integral equations and the theory of tides; the Montyon prize in mechanics to Henri Chipart; the Poncelet prize in mechanics to the late Auguste Boulanger; prizes in navigation to Commandant Charles Lafon for his work *Etudes sur le Ballon captif et les Aéronefs marins*, and to Captain Stanislas Millot for his memoir on *Amarrage, remorquage et mouillage des navires*; the Hebert prize in physics to Professor Edmond Bauer, of the University of Strasbourg, for his work entitled *La Theorie de Bohr*; the Montyon prize in statistics to Alfred Barriol for his statistical work, including his *Théorie et Pratique des*

Opérations financières, and an honorable mention to H. E. Soper for his work entitled *Frequency Arrays*; the Binoux prize for history and philosophy of sciences to Dr. Robert Bouvier, for his work entitled *La Pensée d'Ernst Mach*; the Petit d'Ormoy prize to Professor Elie Cartan, for his mathematical work; the Estrade-Delcros prize to Professor René Baire, for his mathematical work; the Gustave Roux prize to Professor Georges Giraud for his work in automorphic functions; a prize from the Gegner foundation to Professor Augustin Boutaric for his work entitled *La Vie des Atomes*; a prize from the Hirn foundation to Gustave Eiffel for his work in aerodynamics. The Damoiseau prize was not awarded, and the subject, on developments of the work of Poincaré and Liapounoff on figures of equilibrium of a rotating fluid mass (see this BULLETIN, vol. 27, p. 239) was withdrawn.

The scientific prize of the Lasserre Foundation has been awarded to Professor Gaston Julia, of the University of Paris.

The Royal Academy of Madrid announces the following subject for a prize, for which competing memoirs should be presented by October 31, 1924: *A study of the complex collineation in the plane, and its real representation.* The competition is open only to Spanish, Portuguese, and Spanish-American mathematicians.

The Royal Society of London has received a gift from Sir Alfred Yarrow which will be used to endow research professorships, to be awarded only to men of proved ability for independent research. The initial awards of these professorships were to Professor A. Fowler, professor of astrophysics at the Imperial College of Science and Technology, and to Mr. G. I. Taylor, fellow and lecturer in mathematics at Trinity College, Cambridge.

The Cambridge Philosophical Society announces that it will publish shortly, as separate supplements to its PROCEEDINGS, the series of papers by Dr. Niels Bohr, *On the Application of the Quantum Theory to Atomic Structure,* Part I of which has already appeared in volume 13 of the ZEITSCHRIFT FÜR PHYSIK. The translation of Part I will be closely followed by a similar translation af Part II, which it is hoped will appear simultaneously with the German version.

Professor Aimé Cotton, of the Sorbonne, has been elected a member of the Paris Academy of Sciences in the section of general physics, as successor to the late Professor Jules Violle.

Professors Einstein and von Laue have been elected honorary members of the Royal Institution, London.

Professor W. F. Osgood, of Harvard University, has been elected a member of the Leopoldinisch-Carolinische Deutsche Akademie der Naturforscher in Halle.

Dr. Horace Lamb, of Trinity College, Cambridge, has been appointed to give the first Rouse Ball lecture on some subject related to mathematical science (see this BULLETIN, vol. 28, p. 320).

Mr. R. H. Fowler, of Trinity College, Cambridge, has been appointed university lecturer in mathematics.

Dr. C. E. Weatherburn, of Ormond College, Melbourne, has been appointed professor of mathematics at Canterbury University College, Christchurch, New Zealand.

Dr. Ganesh Prasad, dean of the Faculty of Science in the Benares Hindu University, has been appointed Hardinge professor of higher mathematics in the Calcutta University, in succession to Professor C. E. Cullis.

Professors G. D. Birkhoff, of Harvard University, and R. C. Archibald, of Brown University, will lecture at the University of California in the summer of 1924.

Professor H. F. Blichfeldt, of Stanford University, will lecture at Columbia University during the summer session.

Professor Earl Church, of the Pennsylvania Military College, has resigned to take charge of the computation and least squares adjustment of the geodetic survey of the Hawaiian Islands.

Dean James N. Hart, of the University of Maine, who has been head of the department of mathematics for the past thirty years, has been granted a year's leave of absence.

Professor F. W. Owens, of Cornell University, has been granted leave of absence for the second semester of the year 1923–24, and will spend the time in Europe.

Professor C. W. Strom, of Luther College, Decorah, Iowa, has been granted leave of absence for the current year and is studying at the University of Iowa.

Mr. C. C. Wagner, of Allegheny College, has been appointed assistant professor of mathematics at Pennsylvania State College, and is now acting head of the department.

Mr. H. A. Simmons, instructor in mathematics in the University of Michigan, has been appointed assistant professor of mathematics at the University of Pittsburgh.

Associate Professor H. H. Dalaker, of the University of Minnesota, has been promoted to a full professorship of mathematics.

At Fairmount College, Wichita, Kansas, Professor A. J. Hoare has resumed his work as head of the department, having become adjusted to a serious impairment of vision which he suffered four years ago.

Mr. C. S. Whitney has been appointed professor of mathematics and head of the department at the State School of Mines, Miami, Oklahoma.

Professor J. A. Whitted has resigned his position at Hedding College and has been appointed professor of mathematics at Ohio Northern University.

At Simmons College, Abilene, Texas, Professor A. E. Chandler has been made bursar, and Associate Professor J. E. Burnam has been promoted to a full professorship.

Professor Emma K. Whiton, of the University of Redlands, has been appointed professor of mathematics at Mills College.

Professor Auguste Boulanger, directeur des études at the Ecole Polytechnique, died in 1923, at the age of fifty-seven.

Dr. Alexander Gleichen, known for his work in optics, died October 21, 1923, at the age of sixty-one.

The death is announced of Gustaf Eneström, of Stockholm, founder and for eighteen years editor of the BIBLIOTHECA MATHEMATICA.

Professor C. B. Williams, of Kalamazoo College, and Mrs. Williams were guests at the Grand Hotel in Yokohama on the day of the great earthquake. As nothing has been heard from them since that day, the American Embassy reports that they are believed to have lost their lives in the collapse of that hotel. Professor Williams had been a member of the American Mathematical Society since 1895.

Professor J. M. Willard, of Pennsylvania State College, died December 10, 1923, at the age of fifty-eight. Professor Willard had been a member of the American Mathematical Society since 1892.

Professor James Harkness, of McGill University, acting dean of the faculty of arts of that university, died December 7, 1923, at the age of fifty-nine years. Professor Harkness had been a member of the American Mathematical Society since 1891 and had held office as vice-president.

Professor Joseph Lipka, of the Massachusetts Institute of Technology, died January 15, 1924, at the age of forty. Professor Lipka, who received his doctorate from Columbia in 1912, had been on the mathematical staff of the Institute since 1908, with the rank of assistant professor since 1917. He had been requested by the program committee to give a talk at the May meeting of this Society.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BACHMANN (P.). Zahlentheorie. Versuch einer Gesamtdarstellung dieser Wissenschaft in ihren Hauptteilen. Teil 4: Die Arithmetik der quadratischen Formen. 2te Abteilung. Leipzig, Teubner, 1923. 22 + 537 pp.

BAKER (H. F.). Principles of geometry. Volume III: Solid geometry: quadrics, cubic curves in space, cubic surfaces. Cambridge, University Press, 1923. 20 + 228 pp.

BRENKE (W. C.). See DAVIS (E. W.).

CASTLE (F.). Four-figure mathematical tables. London, Macmillan, 1923. 48 pp.

CISOTTI (U.). Lezioni di analisi matematica. 2a edizione. Milano, Tamburini, 1920.

CREAK (T. C.). See CUNNINGHAM (A.).

CUNNINGHAM (A.) and CREAK (T. C.). Fundamental congruence solutions. London, Hodgson, 1923. 18 + 92 pp.

DAVIS (E. W.) and BRENKE (W. C.). The calculus. Revised edition. New York, Macmillan, 1923. 10 + 346 pp.

DOEHLEMANN (K.). Projektive Geometrie in synthetischer Behandlung. 1ter Teil. 5te Auflage. (Sammlung Göschen.) Berlin, de Gruyter, 1922. 132 pp.

FORD (W. B.). A brief course in college algebra. New York, Macmillan, 1923. 7 + 264 pp.

FRAENKEL (A.). Einleitung in die Mengenlehre. 2te erweiterte Auflage. (Courant Series of Monographs.) Berlin, Springer, 1923. 10 + 251 pp.

VAN HAAFTEN (M.). Het Wiskundig Genootschap (Amsterdam). Zijn oudste geschiedenis, zijn werkzaamheden en zijn beteekenis voor het verzekeringswezen. Groningen, Noordhoff, 1923. 163 pp.

IRWIN (J. O.). On quadrature and cubature; or on methods of determining approximately single and double integrals. (Tracts for Computers, No. 10.) London, Cambridge University Press, 1923. 78 pp.

KNOPF (O.). Wahrscheinlichkeitsrechnung. I-II. (Sammlung Göschen.) Berlin, de Gruyter, 1923. 112 + 112 pp.

LEVY (H.). See VON SANDEN (H.).

PICONE (M.). Lezioni di analisi infinitesimale. Volume 1: La derivazione e l'integrazione (parte prima). Catania, Publicazioni del "Circolo Matematico," 1923. 11 + 347 pp.

RAABE (C.). Eine Anwendung der Zetafunktionen auf die Idealzerlegung und Diskriminantenbestimmung in Klassenkörpern. (Dissertation.) Hamburg, Lütcke und Wulff, 1921. 8 pp.

ROBINSON (G.). See WHITTAKER (E. T.).

VON SANDEN (H.). Practical mathematical analysis. With examples by the translator, H. Levy. London, Methuen, 1923. 15 + 195 pp.

SCORZA (G.). Corpi numerici ed algebre. Messina, Principato, 1921. 9 + 451 pp.

STEINHAGEN (P.). Beiträge zur Theorie der konvexen Körper. (Dissertation.) Göttingen, L. Hofer, 1920. 8 pp.

STUYVAERT (M.). Introduction à la méthodologie mathématique. Gand, Van Rysselberghe et Rombaut, et Paris, A. Blanchard, 1923. 8vo. 257 pp.

TROPFKE (J.). Geschichte der Elementar-Mathematik. 2te, verbesserte und vermehrte Auflage. 5ter Band. Berlin, W. de Gruyter, 1923. 185 pp.

DE LA VALLÉE POUSSIN (C. J.). Cours d'analyse infinitésimale. 5e édition refondue. Tome 1. Paris, Hermann, 1923.

VERRIEST (G.). Cours de mathématiques générales. Partie 1: Calcul différentiel. Géométrie analytique à deux dimensions. Louvain, Editions Universitas, et Paris, Gauthier-Villars, 1923. 2 + 337 pp.

DE VRIES (H.). Leerboek der differentiaal- en integraalrekening, en van der differentiaalvergelijkingen. Deel III: Differentiaalvergelijkingen. Groningen, Noordhoff, 1922. 6 + 551 pp.

WEITZENBÖCK (R.). Over de vierde dimensie. Groningen, Noordhoff, 1921. 3 + 411 pp.

WHITTAKER (E. T.) and ROBINSON (G.). A short course in interpolation. London, Blackie, 1923. 8 + 71 pp.

WIELEITNER (H.). Geschichte der Mathematik. II: Von 1700 bis zur Mitte des 19. Jahrhunderts. (Sammlung Göschen.) Berlin, de Gruyter, 1923. 154 pp.

WIJDENES (P.). Middel-algebra. Verschillende onderwerpen uit de lagere algebra en inleiding tot de hoogere algebra. Groningen, Noordhoff, 1921. 3 + 411 pp.

PART II. APPLIED MATHEMATICS

AERONAUTICAL RESEARCH COMMITTEE. Reports and Memoranda, No. 602: Stability of airships, general case; the mathematical theory and its application. London, H. M. Stationery Office, 1923.

ALLIATTA (G.). Missverständnisse. Zu den Grundlagen der Einsteinschen Relativitätstheorie. Zu De Sitters Einwand zu Impulsprinzip. Zum Dopplereffekt. Leipzig, Hillmann, 1923.

APPELL (P.). Traité de mécanique rationnelle. Tome 2. 4e édition. Paris, Gauthier-Villars, 1923.

ARRHÉNIUS (S.). Conférences sur quelques problèmes actuels de la chimie physique et cosmique. Paris, Gauthier-Villars, 1923. 120 pp.

BARKHAUSEN (—.). Elektronen-Röhren. Leipzig, Hirzel, 1923. 6 + 124 pp.

BECQUEREL (J.). Gravitation Einsteinienne. Champ de gravitation d'une sphère matérielle et signification physique de la formule de Schwarzschild. Paris, Hermann, 1923. 32 pp.

BOUASSE (H.) et CARRIÈRE (Z.). Interférences. Paris, C. Delagrave. 1923. 31 + 462 pp.

CARRIÈRE (Z.). See BOUASSE (H.).

CORMAC (P.). A treatise on engine balance. London, Chapman and Hall, 1923. 11 + 151 pp.

ERGEBNISSE der exakten Naturwissenschaften. Herausgegeben von der Schriftleitung der *Naturwissenschaften*. Band I. Berlin, Springer, 1923. 4 + 403 pp.

FABRY (C.). Les applications des interférences lumineuses. Paris, Editions *Revue d'Optique*, 1923. 160 pp.

FAJANS (K.). Radioactivity and the latest developments in the study of the chemical elements. Translated from the fourth German edition by T. S. Wheeler and W. G. King. London, Methuen, 1923.

FISHER (A.). An elementary treatise on frequency curves and their application in the analysis of death curves and life tables. New York, Macmillan, 1922. 16 + 244 pp. $4.00

FÖPPL (A.). Grundzüge der Festigkeitslehre. Leipzig, Teubner, 1923.

FOURNIER (F. E.). Carènes de formes nuisibles ou favorables à leurs grands vitesses; et résistances de l'eau à leur translation. Paris, Gauthier-Villars, 1923. 27 pp.

GLEICHEN (A.). Die Theorie der modernen optischen Instrumente. 2te, neubearbeitete und vermehrte Auflage. Stuttgart, Enke, 1923.

GOODWIN (G.) and others. The mechanical properties of fluids in theory and practice. An account of recent engineering research. London, Blackie, 1923. 16 + 376 pp.

GOTTSCHALK (O.). Elementos del cálcolo mecánico de sistemas elásticos. Buenos Aires, Tomas Palumbo, 1923. 27 pp.

GREGORY (R.). The vault of heaven; an introduction to modern astronomy. London, Methuen, 1923. 7 + 202 pp.

GRUNER (P.). Elemente der Relativitätstheorie. Bern, 1923.

GUNTHER (R. T.). Early science in Oxford. Parts 2, 3, 4: Astronomy, physics, surveying. Oxford, University Press, 1923.

HAGUE, (B.). Alternating-current-bridge methods for the measurement of inductance, capacitance, and effective resistance at low and telephonic frequencies; a theoretical and practical handbook for the use of advanced students. London, Pitman, 1923. 13 + 302 pp.

HAWKINS (C. C.). The dynamo: its theory, design and manufacture. Volume 2. 6th edition, revised throughout and largely rewritten. London, Pitman, 1923. 16 + 322 pp.

HITCHCOCK (F. L.) and ROBINSON (C. S.). Differential equations in applied chemistry. New York, Wiley, 1923. 6 + 110 pp. $1.50

HUDSON (C. W.) and SQUIRE (E. J.). Elements of graphic statics. London, McGraw-Hill, 1923. 8 + 91 pp.

IBBETSON (W. S.). See WALKER (C. A.).

KING (W. G.). See FAJANS (K.).

KIRCHBERGER (P.). Atom- und Quantentheorie. I: Atomtheorie. II: Quantentheorie. (Mathematisch-Physikalische Bibilothek, Band 44 und Band 45). Leipzig, Teubner, 1923.

LAMB (H.). Dynamics. 2d edition. Cambridge, University Press, 1923. 11 + 351 pp.

LANGEVIN (P.). La physique depuis 20 ans. Paris, Doin, 1923. 16mo. 450 pp.

LECORNU (L.). See WORMS DE ROMILLY (P.).

LEUTWILER (O. A.). Problems in machine design. London, McGraw-Hill, 1923. 7 + 133 pp.

LUDEWIG (P.). Die physikalischen Grundlagen des Betriebes von Röntgenröhren mit dem Induktorium. Berlin und Wien, Urban und Schwarzenberg, 1923. 7 + 134 pp.

MCEWEN (B. C.). The properties of matter. London, Longmans, 1923. 6 + 316 pp.

MAINKA (C.). See PREY (A.).

MARGOLIS (L.). See SHADE (S.).

MITCHELL (S. A.). Eclipses of the sun. New York, Columbia University Press, 1923. $3.85

PREY (A.), MAINKA (C.) und TAMS (E.). Einführung in die Geophysik. Berlin, Springer, 1923. 8 + 340 pp.

REY (A.). La théorie de la physique chez les physiciens contemporains. 2e edition. Paris, Alcan, 1923. 9 + 346 pp.

ROBINSON (C. S.). See HITCHCOCK (F. L.).

ROSS (J. F. S.). An introduction to the principles of mechanics. London, Jonathan Cape, 1923. 10 + 400 pp.

SHADE (S.) and MARGOLIS (L.). Mathematics for technical and vocational students. London, Chapman and Hall, 1923. 491 pp.

SOUTHERNS (L.). An outline of relativity. London, Epworth Press, 1923. 47 pp.

SQUIRE (E. J.). See HUDSON (C. W.).

STEINMAN (D. B.). A practical treatise on suspension bridges; their design, construction and erection. New York, Wiley, 1923. 8 + 204 pp.

TAMS (E.). See PREY (A.).

VOGTHERR (K.). Wohin führt die Relativitätstheorie? Kritische Betrachtungen vom physikalischen und erkenntnistheoretischen Standpunkt aus. Leipzig, Hillmann, 1923.

WALKER (C. A.) and IBBETSON (W. S.). Tables and units for engineering students. London, Spon, 1923. 40 pp.

WHEELER (T. S.). See FAJANS (K.).

WORMS DE ROMILLY (P.). Quelques réflexions sur la relativité avec une préface de M. Lecornu. Paris, Hermann, 1923. 59 pp.